Appendix II: The Expected Value and Variance of On-Hand Inventory when there Are no Restrictions on Minimum Buy Quantities

Let:

- \( I \) = On-hand physical inventory
- \( S \) = Order-up-to level
- \( Y \) = Amount of part consumed in first \( L \) weeks of the \( (L + R) \)-week cycle
- \( C_S \) = Cycle stock = stock consumption to date during the \( R \)-week portion of the \( (L + R) \)-week cycle
- \( SS \) = Safety stock

\[
I = S - Y - C_S
\]

\[
E(I) = E\left(\sum_{i=1}^{L} P_i + SS\right) - E\left(\sum_{i=1}^{L} D_i\right) - E(C_S)
\]

\[
E(I) = \sum_{i=1}^{E(L)+R} P_i + SS - \sum_{i=1}^{E(L)} P_i - E(C_S).
\]

We will consider \( C_S \) to be uniformly distributed between 0 and \( \sum_{i=L+1}^{L+R} D_i \). Thus,

\[
E(I) = \frac{E(L)}{2} \sum_{i=E(L)+1}^{E(L)+R} P_i = SS + \frac{RP_R}{2}.
\]

The variance of \( I \) is derived as follows.

\[
V(I) = V(S) + V(Y) + V(C_S)
\]

Even though the \( P_i \) are not all fixed, and hence \( S \) changes every \( R \) weeks, \( S \) is still a constant with respect to the inventory result during the last \( R \) weeks of every \( (L + R) \)-week cycle. Hence, \( V(S) = 0 \).

\[
V(I) = \sum_{i=1}^{L} D_i + V(C_S)
\]

\[
V(I) = \left(\sigma_C^2 + \sigma^2_{L} \right) + V(C_S)
\]

\[
V(C_S) = E\left(\sum_{i=L+1}^{L+R} D_i \right)^2 + V\left(\sum_{i=L+1}^{L+R} D_i \right)
\]

\[
E(C_S) = \frac{D_{L+1} + D_{L+2} + ... + D_{L+R}}{2}
\]
\[
V(\left( C_S \mid D_{L+1}, D_{L+2}, \ldots, D_{L+R} \right)) = V\left( \frac{D_{L+1} + D_{L+2} + \ldots + D_{L+R}}{4} \right)
\]
\[
= \frac{1}{4} V\left( (P_{L+1} + e_{L+1}) + (P_{L+2} + e_{L+2}) + \ldots + (P_{L+R} + e_{L+R}) \right)
\]
\[
= \frac{R \sigma_e^2}{4}
\]
\[
V\left( C_S \mid D_{L+1}, D_{L+2}, \ldots, D_{L+R} \right) = \left( \frac{D_{L+1} + D_{L+2} + \ldots + D_{L+R}}{12} \right)^2
\]
\[
E\left( V\left( C_S \mid D_{L+1}, D_{L+2}, \ldots, D_{L+R} \right) \right)
\]
\[
= E\left( \left( \frac{D_{L+1} + D_{L+2} + \ldots + D_{L+R}}{12} \right)^2 \right) = \frac{1}{12} E(G^2),
\]

where \( G = D_{L+1} + D_{L+2} + \ldots + D_{L+R} \).

\[
E(G^2) = (\sigma_h^2 + \mu^2) = \left( R \sigma_e^2 + \left( \sum_{i=L+1}^{L+R} P_i \right) \right)^2
\]
\[
E\left( V\left( C_S \mid D_{L+1}, D_{L+2}, \ldots, D_{L+R} \right) \right) = \frac{1}{12} \left[ R \sigma_e^2 + \left( \sum_{i=L+1}^{L+R} P_i \right)^2 \right]
\]
\[
V(C_S) = \frac{1}{12} \left[ R \sigma_e^2 + \left( \sum_{i=L+1}^{L+R} P_i \right)^2 \right] + \frac{R \sigma_e^2}{4}
\]

Hence,
\[
V(I) = \sigma_{\Delta h}^2 + \sigma_L^2 \frac{P_L^2}{1} + \frac{1}{12} \left[ R \sigma_e^2 + \left( \sum_{i=L+1}^{L+R} P_i \right)^2 \right] + \frac{R \sigma_e^2}{4},
\]

where \( P_L \) is the average of the plan over the \( L \)-week period immediately before the \( R \)-week period in question.