

Affine and Projective Geometries

A Tutorial

Hans J. Wolters

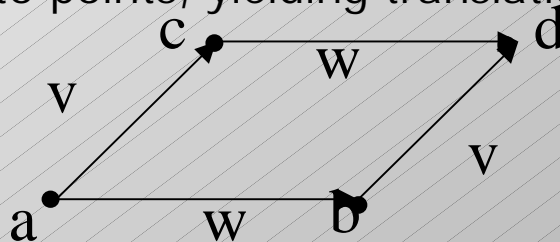
Visual Computing Department

Hewlett-Packard Laboratories

Palo Alto, CA

Affine Spaces

- Affine spaces are the spaces we are mostly familiar with in 3D graphics and modeling:
- The elements are **points** and **vectors**.
- There is no preferred origin for the coordinate system in affine space, hence all our subsequent definitions must be independent of the particular choice of coordinate system or origin more precisely.
- Not all operations are valid: If **a** and **b** are points then $\mathbf{b} - \mathbf{a} = \mathbf{v}$ is a vector, however $\mathbf{a} + \mathbf{b}$ is not defined, one can not simply add points, however one can add and subtract vectors yielding an other vector and add vectors to points, yielding translations.



Two points describe one unique vector, but one vector describes infinitely many point pairs

Barycentric combinations and affine maps

- Addition of points is not coordinate-independent however we can form **barycentric combinations** of points. These are combinations

$\sum_{j=0}^n \alpha_j \mathbf{b}_j$ where the weights α_j sum up to 1.

- Note: $\sum_{j=0}^n \alpha_j \mathbf{b}_j = \mathbf{b}_o + \sum_{j=1}^n \alpha_j (\mathbf{b}_j - \mathbf{b}_o)$. This is the sum of a point and vector(s)
- Example is the barycentric combination of three points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ forming a triangle. Weights $1/3$, yield the centroid.
- Affine maps are maps which leave these barycentric combinations invariant. That means that the ratio of points on a line is preserved, the same is true triangles. If \mathbf{g} is the centroid of a triangle formed by $\mathbf{a}, \mathbf{b}, \mathbf{c}$ then $A\mathbf{g}$ is the centroid of the triangle formed by $A\mathbf{a}, A\mathbf{b}, A\mathbf{c}$.

Affine Maps - continued

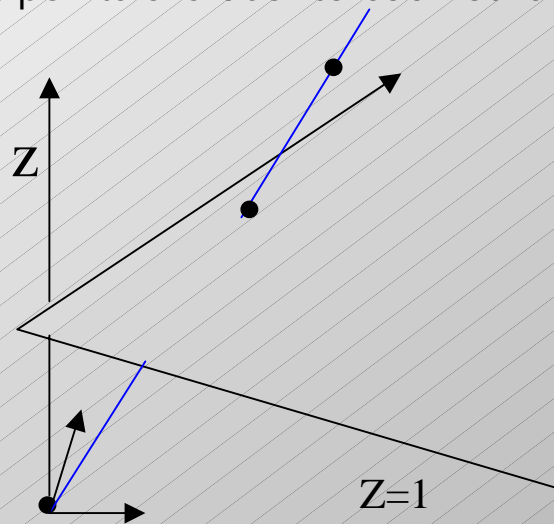
- How are affine maps characterized: An affine map M in n -dim space consists of a $n \times n$ matrix A and a (translation vector) \mathbf{v} : $M\mathbf{x} = A\mathbf{x} + \mathbf{v}$
- It is easy to show that the above mapping leaves barycentric combinations invariant.
- The most canonical affine maps are: translation ($A=I$), scaling, rotation, shear. Out of these we can build more complicated maps.
- If $\mathbf{A}^T \mathbf{A} = \mathbf{I}$ then we talk about **Euclidean maps** or **rigid body motions**.
- It is noteworthy that linear interpolation maps the straight 1 D line affinely into n space. Hence linear interpolation is affinely invariant.

Connections to Euclidean vector space.

- The canonical vector space in \mathbf{R}^n endowed with the familiar dot product allows us to measure angles and distances. This space is known as the **Euclidean vector space**. Here we have still the cartesian coordinate system as our coordinate system of choice.
- If we endow our affine space with the same dot product for vectors, we can measure distances and angles as well. We could call this an **Euclidean affine space**.
- Not all affine maps leave angles and distances invariant, the ones which do are called **isometries**. However, each isometry is affine.
- **NOTE:** It is common to call the Euclidean affine space just Euclidean space or affine space. Note however that in the mathematical sense there are differences.

Projective Geometry

- So far we have dealt with Euclidean geometry. For dealing with how we see things the projective viewpoint is more advantageous.
- Some features:
 - Two lines always converge
 - Lines and points are dual to each other



All points on the line are
indistinguishable to observer at
origin

Points and Lines in Homogeneous Coordinates

- Restricting ourselves to the projective plane the previous picture motivates the following definition:
- The projective plane \mathbf{P}^2 consists of points $\mathbf{p} = [p_1, p_2, p_3]^T$ and lines $\mathbf{L} = [l_1, l_2, l_3]$

Two points \mathbf{p} and $s\mathbf{p}$ denote the same location, two lines \mathbf{L} and $s\mathbf{L}$ denote the same line. This shows immediately that the three coordinates are unique up to a scaling factor, it is common to scale the third coordinate to 1 if it is not equal 0.

- A point \mathbf{p} is on a line \mathbf{L} if the dot product \mathbf{Lp} vanishes.

Duality of Points and Lines

- Points and lines are dual to each other in projective space:
 - if \mathbf{p} and \mathbf{q} are points then the cross product $\mathbf{L} = \mathbf{p} \times \mathbf{q}$ denotes the line defined by the two points.
 - If \mathbf{L} and \mathbf{M} are lines, then the cross product $\mathbf{p} = \mathbf{L} \times \mathbf{M}$ denotes the point where the two lines intersect.
 - Clearly if points \mathbf{p} and \mathbf{q} define a line then if for a third point \mathbf{x} we have $[\mathbf{p} \times \mathbf{q}] \cdot \mathbf{x} = 0$, then \mathbf{x} lies on the same line. In short $\det[\mathbf{p}, \mathbf{x}, \mathbf{q}] = 0$.
 - For lines $\det[\mathbf{M}, \mathbf{N}, \mathbf{L}] = 0$ implies that the three lines \mathbf{M}, \mathbf{N} and \mathbf{L} are concurrent.
 - Two parallel lines $\mathbf{L} = [l, m, k]$ and $\mathbf{M} = [l, m, n]$ intersect in a point $[p, q, 0]^T$. This is the point at infinity.
- Some more interesting properties can be discussed, I refer to
Farin, G., NURBS Curves and Surfaces, Chapter 1 and 2.

Affine and Projective Plane

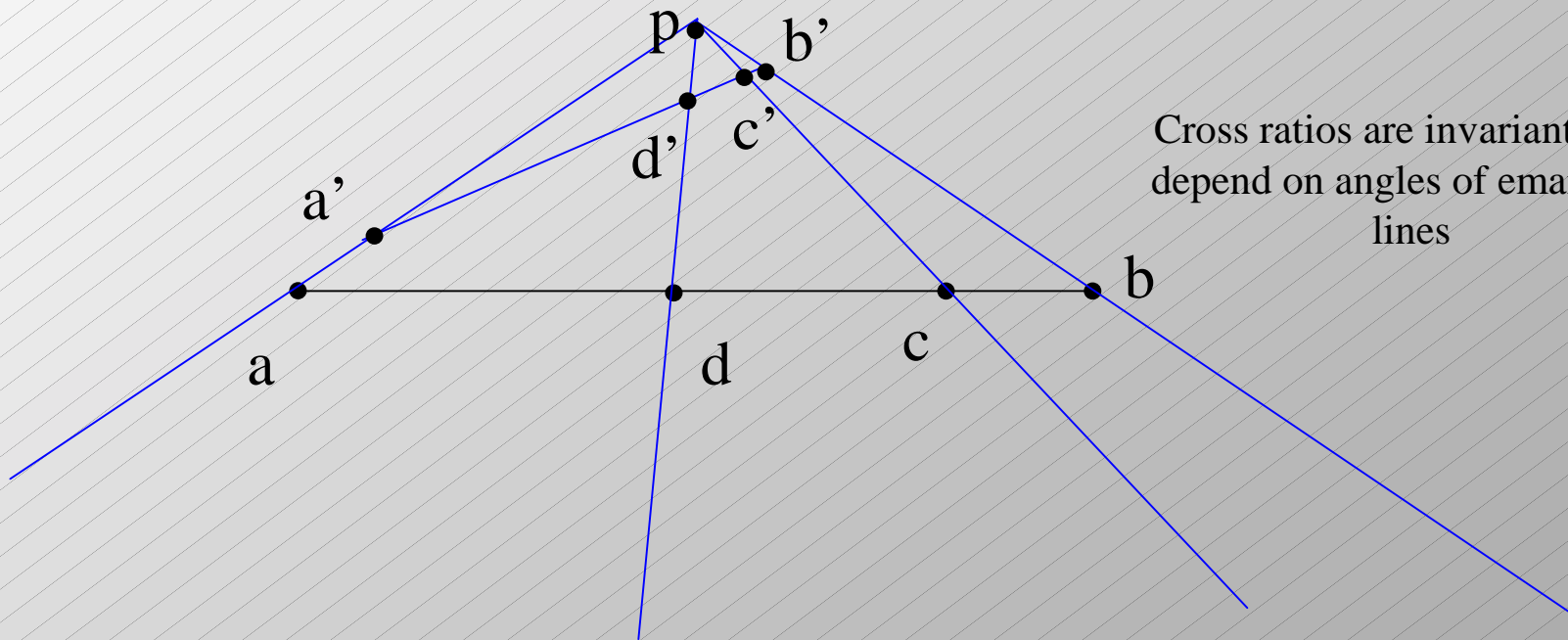
- Figure showed that projective points were obtained as projections of affine 3 space into plane $z=1$. Hence each projective points can be considered as affine point with coordinates $[p_1, p_2, 1]^T$. Hence each projective point $\mathbf{p} = [x, y, z]^T$ can be identified with an affine point $[x/z, y/z]$.
- In affine space the difference of two points \mathbf{p} and \mathbf{q} is a vector. In projective space with $\mathbf{a} = [a, b, 1]^T$ and $\mathbf{b} = [c, d, 1]^T$ the difference is $[a-c, b-d, 0]^T$. Hence vectors in affine space correspond to points at infinity in projective space.
- Unifying the affine and projective view leads to different algebra such as Grassmann algebra or Clifford algebra.

Collineations

- The collineations are the projective counterpart to affine maps.
- Collineations map the projective plane onto itself with the following properties:
 - (collinear) points are mapped to (collinear) points
 - (concurrent) lines are mapped to (concurrent) lines
 - redundant: pencils are mapped to pencils
- Collineations are given by a 3×3 matrix which is nonsingular. The matrix is unique up to a constant scale factor. Hence 4 point pairs define a collineation.

Cross Ratios

- Ratios are invariants in affine space. What is the counterpart here: **cross ratios**: Cross ratios are a ratio of ratios: Let a, b, c, d be points, where $c = sa+tb$ and $d = ua+vb$, then $cr = (t/s)/(v/u)$.



Cross ratios are invariant, only depend on angles of emanating lines