Private Database Queries Using Quantum States with Limited Coherence Times

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Abstract

We describe a method for private database queries using exchange of quantum states with bits encoded in mutually incompatible bases. For technology with limited coherence time, the database vendor can announce the encoding after a suitable delay to allow the user to privately learn one of two items in the database without the ability to also definitely infer the second item. This quantum approach also allows the user to choose to learn other functions of the items, such as the exclusive-or of their bits, but not to gain more information than equivalent to learning one item, on average. This method is especially useful for items consisting of a few bits by avoiding the substantial overhead of conventional cryptographic approaches.

1 Introduction

Quantum information processing provides potentially significant performance improvements over conventional techniques. One example is quantum computation with its ability to rapidly solve problems, such as factoring, which appear to be otherwise intractable. However, implementing machines with enough bits and coherence time to solve computational problems difficult enough to be of practical interest is a major challenge. Another application, quantum cryptography, is feasible today for exchanging keys over distances of tens of kilometers. A third application area is to quantum economic mechanisms, which can offer benefits with only a few qubits which should be feasible to implement relatively soon.

Early quantum information technology is likely to be characterized by few operations before decoherence, limited ability to store coherent states and communication involving limited entanglement, particularly restricted to pairwise entangled states. Such limited technology will not provide significant computational advantages over conventional techniques. Nevertheless, limited quantum
information capabilities provide alternative economic mechanisms for situations benefiting from correlated behaviors among participants and the information security provided by quantum states. Examples include provisioning public goods [5, 29], coordinating random choices without communication [12, 21] and auctions [10, 8]. These economic methods can function with limited quantum information technology, but do not require it. In contrast, the private database query method described in this paper relies on the difficulty of maintaining coherence for long times to be economically viable.

Privacy-enhanced mechanisms can be instrumental in encouraging beneficial transactions in situations where participants face economic or other costs if their information is revealed to others. Examples based on cryptographic methods include allowing long-term surveys on sensitive social or medical topics [11] and auctions [23]. The problem treated in this paper arises when a user wishes to purchase some information from a vendor without revealing what information is desired, but also not paying for additional information (e.g., purchasing the entire database).

In the remainder of this paper, we first describe the private database query problem in its most basic case: selecting one of two bits. We then prove, by applying the generalized entropic uncertainty relations proven in [17], that the user can learn at most one bit under the assumption that maintaining coherence beyond a limited time is not possible with the technology available or too costly compared to the economic value of the information. We then briefly consider the generalization to larger databases with many bits of information on multiple items, and conclude with a discussion of possible applications.

The mechanism provided here differs from providing private information exchange with cryptographic methods (i.e., learning exactly one bit and nothing more), or quantum attacks relying on more advanced quantum technology such as creating and storing entangled states until completing the protocol [16]. This illustrates the importance of understanding plausible capabilities of adversaries, especially in the context of an emerging technology where advanced capabilities are likely to be too expensive (or not available) to justify the cost compared to, say, just purchasing the additional information from the vendor.

The problem we deal with in this paper is known as symmetrically private information retrieval (SPIR) or oblivious transfer (OT) in the cryptography community. We refer to [6] for an excellent survey of the subject. In the classical computation model, one can achieve computationally secure SPIR with a single server under appropriate computational hardness assumptions [15, 3, 22, 20]. When there are multiple servers which do not communicate with each other, one can design information theoretically secure SPIR [7]. The emphasis in those studies is on reducing the communication complexity.

Quantum channels allow reducing the communication complexity in the case of more than one server [14]. The method studied in this paper only uses one way communication from the vendor to the user. Therefore, the privacy of the user is guaranteed, and only the privacy of the vendor is of concern. In such
a model, if the length of the database is \( n \) bits, the vendor must communicate \( \Omega(n) \) bits of information, and it is impossible to guarantee the vendor’s privacy in the information theoretic sense even with prepared entangled states between the vendor and the user \([23, 1]\). In this paper, we show that when the user can only store entangled states for a short time, we may achieve information theoretic SPIR. This has similar flavor to the previous study in which the user is memory constrained \([2]\).

2 One out of two exchange

In this section, we consider the case when the vendor has a database of two items, each with \( m \) bits, and wishes to deliver one and only one item to the user according to the user’s private choice, which is not revealed to the vendor.

The vendor picks an encoding for the value of each message, represented as \( 2^m \) bits. These bits are sent to the user who then must measure them, in some choice of basis, and wait for the vendor to announce the choice of encoding. Knowing the measurement outcome and the encoding allows the user to determine all \( m \) bits of one item. To prevent the user from learning both messages (with probability 1), it is important that the user’s measurement take place before the vendor announces the encoding – otherwise the user could invert the operator producing the announced encoding and recover both items. The protocol ensures this based on a limited coherence time of the available technology – the vendor simply waits until well past the coherence time before announcing the encoding choice.

2.1 Single bit exchange

As an illustrative example, consider a database with two items, each with one bit of information (e.g., a recommendation to buy or sell a company’s stock). The vendor selects two maximally incompatible measurement bases:

\[
|0\rangle, |1\rangle
\]

and

\[
\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)
\]

The vendor encodes the database into a superposition of the two bits such that measurement in these bases reveals the bit value corresponding to the first or second item in the database, respectively.

To do this, the vendor encodes the value for each item in two bits, randomly choosing one of two encodings. For the first encoding, the vendor specifies the first and second bits using the first and second of these bases, respectively. The superposition sent to the user is one of the following, according to whether the
database values are 00, 01, 10 or 11, respectively:

\[
\begin{align*}
\frac{1}{\sqrt{2}} & (|00\rangle + |01\rangle) \\
\frac{1}{\sqrt{2}} & (|00\rangle - |01\rangle) \\
\frac{1}{\sqrt{2}} & (|10\rangle + |11\rangle) \\
\frac{1}{\sqrt{2}} & (|10\rangle - |11\rangle)
\end{align*}
\]

The second encoding specifies the bits using the same bases, but forms the superposition of the two bits in the opposite order. The corresponding superpositions for database values 00, 01, 10 or 11, respectively, are:

\[
\begin{align*}
\frac{1}{\sqrt{2}} & (|00\rangle + |10\rangle) \\
\frac{1}{\sqrt{2}} & (|00\rangle - |10\rangle) \\
\frac{1}{\sqrt{2}} & (|01\rangle + |11\rangle) \\
\frac{1}{\sqrt{2}} & (|01\rangle - |11\rangle)
\end{align*}
\]

The user can obtain the single bit associated with either item by electing to measure in one of the two announced bases.

For example, suppose the bit values in the database are 01. If the vendor chooses the first encoding, the user receives the state \( |\psi\rangle = (|00\rangle - |01\rangle)/\sqrt{2} \) as represented in the first basis. Measurement in the first basis gives either 00 or 01, and the user will use this measured result to learn the value for the first item is 0 once the vendor announces the choice of encoding. Expressed in the second basis, the state \( |\psi\rangle \) is \( (|01\rangle + |11\rangle)/\sqrt{2} \). Thus choosing to measure in the second basis gives the user either 01 or 11, which specifies the value of the second item is 1.

On the other hand, if the vendor chooses the second encoding, the user receives the state \( |\phi\rangle = (|00\rangle - |10\rangle)/\sqrt{2} \). Measurement in the first basis gives either 00 or 10, and the user will learn the value for the first item is 0 (based on the 2nd bit of either of these outcomes) once the vendor announces the choice of encoding. Expressed in the second basis, the state \( |\phi\rangle \) is \( (|10\rangle + |11\rangle)/\sqrt{2} \). Thus choosing to measure in the second basis gives either 10 or 11, indicating the value of the second item is 1 (based on the 1st bit of either of these outcomes).

In either of these cases, the measured outcome in one basis gives no information on the value of the item associated with the other basis.

The user could instead choose any other basis for the measurement, or even use different bases for the two bits. Such choices can reveal a function of the
bit values for both items, e.g., their exclusive-or. That is, the user could learn whether both bits in the database are the same without learning anything about their values (e.g., which could be used as a recommendation regarding a joint derivative instrument on both companies, such as a recommendation of whether they will both change in the same direction). However, as described below, no choice of basis allows the user to learn more than one bit of information about the database, on average. Thus in contrast to conventional (i.e., cryptographic) methods for private database query, the user has a wider set of options than just picking one of the database bits to learn.

If the user knows the encoding used by the vendor, then instead of selecting a basis to measure the states, the user could apply the inverse of the 2-qubit encoding operation to produce values for both bits in the database upon final measurement. However, doing this using the wrong encoding gives no information about either bit. Thus with limited coherence time, the vendor simply waits longer than that interval before announcing the choice of encoding. In that case the user must make a measurement before learning the encoding. After learning the encoding, the user would then know whether both or neither of the bits are revealed, but would no longer have the original quantum state. Each alternative occurs with probability 1/2 so, on average, only one bit of information from the database is revealed. The question now is whether there exists some basis for the user to learn more than one bit of information. We show this is impossible in the following sections.

2.2 General formulation

Suppose the vendor has two one-bit values, \( d_0 \) and \( d_1 \). These two values specify the vendor’s database state \( d = |d_0d_1\rangle \). The vendor has two encoding operators acting on the two bits of the database, specified as \( 4 \times 4 \) encoding matrices \( E_0 \) and \( E_1 \). These matrices are unitary, i.e., \( E_i^\dagger = E_i^{-1} \) where \( E_i^\dagger \) denotes the adjoint (i.e., complex conjugate transpose) of \( E_i \). The vendor announces these two operators to the user.

The vendor then randomly picks one of the two encodings, say \( E_i \), and sends the state \( e = E_i d \) to the user. That is, \( e \) is column \( d \) of the encoding matrix \( E_i \). The user selects a measurement basis, given as the columns of a unitary matrix \( M \). This measurement is a standard projective or von-Neumann measurement. We consider the more general POVM case in Section 2.4. Thus the user measures the state \( Me \), obtaining outcome \( j \) with probability \( P(j|d,i) = |(Me)_j|^2 = |(ME_i)_{j,d}|^2 \), conditioned on the vendor’s choice of encoding \( i \) and the value of the database \( d \). The user is free to choose any basis, but selects \( M \) and performs the measurement without knowledge of which encoding the vendor selected. After the measurement, the vendor announces the encoding choice \( i \).

From the observation \( j \) and choice of encoding \( i \) the user can use Bayes’ theorem to obtain a posterior probability distribution over the values of the
database. Specifically,
\[
P(d|j, i) = \frac{P(j|d, i) P(d)}{\sum_{d'} P(j|d', i) P(d')} = \frac{P(j|d, i)}{\sum_{d'} P(j|d', i)}
\]  
(3)

where \(P(d)\) is the prior distribution on the database, and the last expression follows from our assumption this prior is uniform, i.e., \(P(d) = 1/4\) is independent of \(d\). The sum in the denominator is \(\sum_{d'} |(ME_i)_{j,d}|^2 = 1\) since the matrix \(ME_i\) is unitary. Thus the user’s knowledge of the database, given by the distribution on the values \(d\), is \(P(d|j, i) = |(ME_i)_{j,d}|^2\).

The uncertainty in the user’s knowledge of the database after this procedure is the entropy of this distribution, \(h_{j,i} = H\{P(0|j, i), \ldots, P(3|j, i)\}\) where for a probability distribution \(P = \{p_0, \ldots, p_{n-1}\}\), \(H(P) = -\sum k p_k \log p_k\) is its entropy\(^1\).

Since each encoding choice is equally likely and not known to the user at the time of measurement, on average the user’s remaining uncertainty about the database is \(h_j = (h_{j,0} + h_{j,1})/2\). The entropy of the prior distribution, \(H(1/4, 1/4, 1/4, 1/4) = 2\) so the amount of information the user gains, averaged over the vendor’s choice of encoding, is \(2 - h_j\). That is, this is the expected value of the reduction in the user’s uncertainty (i.e., entropy) of the inferred distribution over the database items when the vendor chooses each encoding with equal probability.

To bound the user’s information gain, we need a lower bound on \(h_j\). We obtain such a bound as a special case of the generalized entropic uncertainty relations [17]. For a complex unit vector \(u = (u_0, \ldots, u_{n-1})^\dagger \in \mathbb{C}^n\), define \(H_2(u) = H\{|u_0|^2, \ldots, |u_{n-1}|^2\} = -\sum p_i \log p_i\) where \(p_i = |u_i|^2\). For any two \(n \times n\) matrices \(A\) and \(B\), let \(c(A, B) = L_\infty(AB)\), where \(L_\infty(X) = \max_{i,j} |X_{ij}|\). In [17], it is shown that

\[
H_2(Au) + H_2(Bu) \geq -2 \log c(A, B).
\]

In particular, suppose \(AB\) is a Hadamard matrix — an \(n \times n\) matrix \(W = (w_{ij})\) is a Hadamard matrix if \(W\) is unitary and if \(|w_{ij}| = 1/\sqrt{n}\). Then \(c(A, B) = 1/\sqrt{n}\) and \(H_2(Au) + H_2(Bu) \geq \log n\).

To apply this bound to our case, let \(u^\dagger = \hat{\epsilon}_j^\dagger ME_0\) and \(v^\dagger = \hat{\epsilon}_j^\dagger ME_1\) where \(\hat{\epsilon}_j\) is the unit vector with \((\hat{\epsilon}_j)_j = 1\) and \((\hat{\epsilon}_j)_k = 0\) for \(k \neq j\). Then \(u_j = (ME_0)_{j,d}\) so \(P(d|j, 0) = |u_d|^2\) and, similarly, \(P(d|j, 1) = |v_d|^2\). We have \(v = E_1^\dagger E_0 u\), so taking \(A\) to be the identity matrix and \(B = E_1^\dagger E_0\) we have

\[
H_2(Au) + H_2(Bu) = H_2(u) + H_2(v) = h_{j,0} + h_{j,1}\]

\(^1\)Throughout the paper, all the logarithms are base 2 unless explicitly stated.
and so \( h_j \geq -\log c(I, E_1^j E_0) \) from Theorem 2.1. This bound is independent of \( j \), i.e., the particular outcome the user measures.

Thus if the vendor’s choice of encodings have \( E_1^j E_0 \) a Hadamard matrix, and the user starts with a uniform prior distribution for the \( n \) states, with \( H = \log n \), the average information gain for the user is

\[
I_{\text{gain}} = \log n - h_j \leq \log n - \log \frac{1}{\sqrt{n}} = \frac{1}{2} \log n
\]  

(4)

The encoding matrices corresponding to the example in Section 2.1 are \( E_0, E_1 \) equal to

\[
\begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\]

respectively. The encoding \( E_1 \) includes the permutation of the bits of the database. \( E_1^j E_0 \) has all entries equal to \( \pm 1/2 \), i.e., is a \( 4 \times 4 \) Hadamard matrix. Thus, from Eq. 4 with \( n = 4 \), the average amount of information the user gains is at most one bit. Moreover, this bound is tight: the measurement examples described in Section 2.1 provide the user one bit of information.

Note the bound applies to the expected information gain, averaged over the choice of encodings. As described in Section 2.1, the user could choose to invert one of the encodings, e.g., take \( M = E_0^\dagger \). If the user correctly guesses the encoding used by the vendor, this procedure gives the user both bits of the database, i.e., \( h_{j,0} = 0 \). But if instead the vendor selected \( E_1 \), the measured outcome is completely uninformative, with \( h_{j,1} = 2 \). So the average of these equally likely possibilities, \( h_j = 1 \), satisfies the bound.

Thus we establish that a vendor, using maximally incompatible bases for encoding bits, can arrange for the user to learn no more than one bit of information, on average, about the two bits in the database.

### 2.3 Multiple-bit exchange

Theorem 2.1 applies to vectors with any number of components \( n \), not just \( n = 4 \) as used for the single-bit database items illustrated in Section 2.1. It is well known that for any \( m > 0 \), there exist \( 2^m \times 2^m \) Hadamard matrices. The standard construction is \( W_{2^m} = W_2 \otimes W_2 \otimes \cdots \otimes W_2 \), where \( W_2 = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 \\
-1
\end{pmatrix}, W_2 \). Hence, we can generalize the above scheme to exchanging two items, each with \( m \) bits.

Suppose that the vendor has two messages \( d_0, d_1 \), each with \( m \) bits. Let \( \ell = 2^m \). Pick unitary \( \ell \times \ell \) matrices \( A_0 \) and \( A_1 \) such that \( A_0^\dagger A_1 \) is Hadamard.
Thus $A_1^\dagger A_0$ is also Hadamard. Let $C_0 = A_0 \otimes A_1$, $C_1 = A_1 \otimes A_0$ and $P$ be
the permutation matrix to reverse the order of the items, i.e., mapping $d_0d_1$ to $d_1d_0$. The vendor’s encoding operators are then $E_0 = C_0$ and $E_1 = C_1 P$. That is, for the first encoding, the vendor sends the column of $C_0$ indexed by $d_0d_1$ and for the second encoding sends the column of $C_1$ indexed by $d_1d_0$, as in the single bit example. With this formulation, the discussion of Section 2.2 applies directly to this case.

Let $M_j = A_0^\dagger \otimes A_0^\dagger$ for $j = 0, 1$. It is easy to verify that $M_j C_i$ is the tensor
product of two $\ell \times \ell$ matrices with the $((j - i) \mod 2)$-th component being the
identity matrix. This is coincident with the position of item $d_j$ in the permuted
string when the vendor uses the encoding matrix $C_i$. Therefore, if the user
applies the measurement $M_j$ and later receives the value of $i$, he learns
$d_j$ with probability 1, regardless of which $C_i$ the vendor uses.

We observe that

$$E_1^\dagger E_0 = P^\dagger C_1^\dagger C_0 = P^\dagger \left( (A_1^\dagger A_0) \otimes (A_0^\dagger A_1) \right),$$

which is a $n \times n$ Hadamard matrix where $n = 2^{2m} = \ell^2$ is the number of possible
configurations for the database. Thus, by Theorem 2.1, we have that no matter
which measurement the user applies, if the prior distributi on of the messages is
uniform, then the user can learn at most $\log \ell = m$ bits in expectation.

This construction extends the result of Section 2.2 to show a vendor can
arrange for the user to learn no more than $m$ bits of information, on average,
about the two $m$-bit items in the database.

To make it easy to encode and measure, we can let $A_0, A_1$ take the form of
tensor product of $m$ $2 \times 2$ matrices so that the encoding and measurement can
be done by single bit operations. One choice with this property is $A_0 = I$ and
$A_1 = W_\ell$.

There are two drawbacks in the above scheme. First, although in expectation
the user learns $m$ bits, his chance of learning both messages is $1/2$ by guessing
right which encoding the vendor uses. One way to reduce this probability is to
split $d_0 = d_{01} \oplus d_{02} \oplus \cdots \oplus d_{0r}$ and $d_1 = d_{21} \oplus d_{22} \oplus \cdots \oplus d_{2r}$, where $\oplus$ represents
bitwise xor. Then the vendor applies the above scheme to the pairs $(d_{1i}, d_{2i})$ for
$1 \leq i \leq r$. The honest user can still learn the message according to his choice.
But if the user wants to learn both messages, he will have to guess right for each
$1 \leq i \leq r$ which happens with probability $1/2^r$.

The other property of our scheme is that the user may choose to learn any
$m$ bits in the combined message $c = d_0d_1$, rather than just all $m$ bits of one of
the two items. This would be useful if the vendor would like to give the user
the freedom to decide which bits to learn. On the other hand, this property
may be considered as a violation of security, for example, when the vendor
would like the user to learn only one item but not even partial information
about the other item. This is of course impossible to achieve in the information
theoretic sense. But the following simple scheme may prevent the user from
learning individual bits of the original message. In the modified scheme, the
vendor treats each message as an element in Galois field \( GF(2^m) \) and picks two random \( a \neq 0, b \in GF(2^m) \) and applies the above protocol to the messages \( d'_0 = a \cdot d_0 + b \) and \( d'_1 = a \cdot d_1 + b \), where \( \cdot, + \) are the arithmetics in \( GF(2^m) \). At the end, the vendor announces \( a, b \) together with the encoding scheme he uses. Clearly, an honest user is still able to recover the message according to his choice. On the other hand, if the user only learns, for example, a constant fraction of the message \( d'_0 \) before he knows \( a, b \), intuitively, it is unlikely that the user can determine any individual bit of \( d_0 \).

2.4 Generalized measurements

Our discussion considered users making conventional projective measurements on the states they receive from the vendor. A more general possibility is positive operator valued measurements (POVM) \[25\]. In some cases, such measurements can distinguish quantum states with higher probability than any projective measurement. In this section, we briefly describe these measurements and show they provide no benefit in the context of the two-item database described above.

A POVM consists of a set of \( N \) operators \( \{ R_1, \ldots, R_N \} \) in an \( n \)-dimensional Hilbert space. The operators are not necessarily Hermitian, orthogonal or invertible, and \( N \) may be larger than \( n \). These operators satisfy

- completeness
  \[
  \sum_j R_j^\dagger R_j = I
  \]

- nonnegativity: for every vector \( x \)
  \[
  x^\dagger R_j^\dagger R_j x \geq 0
  \]

For a system in a pure state \(|\psi\rangle\), measurement gives one of the outcomes \( j = 1, \ldots, N \), with probability for outcome \( j \) equal to \( P(j) = \langle \psi | R_j^\dagger R_j |\psi\rangle \). This probability is nonnegative due to the nonnegativity condition, and \( \sum_j P(j) = 1 \) due to the completeness condition. The state after measurement is

\[
\frac{1}{\sqrt{P(j)}} R_j |\psi\rangle
\]

**Example.** For a projection measurement, the POVM consists of \( N = n \) orthogonal projection operators: \( R_j = |e_j\rangle\langle e_j| = e_j e_j^\dagger \) where \( e_j \) is the \( j^{th} \) unit basis vector of the measurement. The probability to observe outcome \( j \) for state \(|\psi\rangle\) is \( \langle\psi| e_j\rangle \langle e_j|\psi\rangle \) or \( |\langle e_j|\psi\rangle|^2 \), in which case the resulting state is \( |e_j\rangle\langle e_j|\psi\rangle / |\langle e_j|\psi\rangle| \), which is \(|e_j\rangle\) up to a phase factor.

In our context of two database items, each with \( m \) bits, the Hilbert space has dimension \( n = 2^{2m} \). The vendor picks one of two encoding operators, \( E_0 \) or \( E_1 \), and sends the state \( e = E_i d \) to the user. Suppose the user applies the POVM
\{ R_j \}. With probability \( P(j|d, i) = (E_i^d R_j^i R_j E_i)_{dd} \) the user observes outcome \( j \) conditioned on the vendor’s encoding choice \( i \) and value of the database \( d \).

Eq. 3 gives user’s inference of the distribution of database values from the outcome of the measurement and the vendor’s announced choice of encoding. The sum in the denominator of Eq. 3, \( \sum_d' P(j|d', i) \), is \( \text{Tr}(E_i^d R_j^i R_j E_i) \) which equals \( \text{Tr}(R_j^i R_j) \) since \( E_i \) is unitary. In particular, this sum is independent of the vendor’s encoding choice \( i \). We denote this sum by \( s^2 \) with \( s > 0 \). Eq. 3 then gives

\[
P(d|j,i) = \frac{P(j|d,i)}{s^2} = (E_i^d S^i S E_i)_{dd}
\]

where \( S = R_j/s \) so \( \text{Tr}(S^i S) = 1 \).

Let \( A = SE_0 \). Then \( SE_1 = AU \) where \( U = E_0^i E_1 \) is a Hadamard matrix for the choice of encodings described above. Thus \( P(d|j,0) = (A^i A)_{dd} \) and \( P(d|j,1) = (U^i A^i A U)_{dd} \).

By orthogonalizing \( A \), we have \( A^i A = \sum_r \lambda_r v_r v_r^\dagger \) where \( \lambda_r \geq 0 \), \( \sum_r \lambda_r = 1 \), and the \( v_r \)'s are mutually orthogonal unit vectors. For each \( r \) we define two probability distributions over the values \( d \) in the database: \( p_{d}^{(r)} = |(v_r)_d|^2 \) and \( q_{d}^{(r)} = |(U^i v_r)_d|^2 \). Then \( P(d|j,0) = \sum_r \lambda_r p_{d}^{(r)} \) and \( P(d|j,1) = \sum_r \lambda_r q_{d}^{(r)} \).

By Theorem 2.1, we have \( H(p^{(r)}) + H(q^{(r)}) \geq \log n \) because \( U \) is a Hadamard matrix. Using this bound and the convexity of the entropy function, we have

\[
H(\{ P(d|j,0) \}) + H(\{ P(d|j,1) \}) = H\left( \sum_r \lambda_r p^{(r)} \right) + H\left( \sum_r \lambda_r q^{(r)} \right)
\geq \sum_r \lambda_r \left( H(p^{(r)}) + H(q^{(r)}) \right)
\geq \log n \sum_r \lambda_r = \log n.
\]

Thus Eq. 4 applies to this POVM, giving the same bound as for the projective measurements considered above. Since projective measurements can achieve the lower bound, we see POVM provides no advantage for the user in this context.

### 3 One out of \( k \) exchange

The previous section considered a database of two items. We showed how the user could privately learn one item, and no more, when the vendor selects randomly from two encodings related by a Hadamard matrix and relies on limited coherence time to force the user to make a measurement before the choice of encoding is announced.

A natural extension is to a database with \( k \) items, each consisting of \( m \) bits. The mechanism would then allow the user to privately learn a limited number
of bits, on average, as well as provide an opportunity to learn all the bits of any one item with probability one. Table 1 summarizes our notation.

The scheme is similar to the case of $k = 2$. The vendor chooses $k$ encodings $C_0, \cdots, C_{k-1}$ in the joint space of $d_0, \cdots, d_{k-1}$, each $m$-bit long. The vendor chooses one encoding randomly among those candidate encodings to encode the items and send to the user. The user measures the state before the vendor announces the encoding. We would like the following properties hold:

1. For each $0 \leq i \leq k-1$, there exists a measurement $M_i$ such that the user learns $d_i$ for sure after the vendor announces the encoding.

2. For any measurement $M$ made before the vendor announces the encoding, including POVM, the user can only learn at most $m$ bits when averaged over the vendor’s choice of encoding.

As an extension to $k = 2$ case, we consider the following encoding scheme. Let $A_0, A_1, \ldots, A_{k-1}$ be $\ell \times \ell$ unitary matrices. Let

$$C_i = A_i \otimes A_{i+1} \otimes \cdots \otimes A_{k-1} \otimes A_0 \cdots \otimes A_{i-1}. \quad (7)$$

For each $0 \leq i \leq k-1$, we use $C_i$ to encode the concatenated string $c_i = d_id_{i+1} \cdots d_{k-1}d_0 \cdots d_{i-1}$, so the corresponding encoding matrix is $E_i = C_iP_i$ where $P_i$ is the permutation giving this reordering of the string. Let $M_j = A_j^\dagger \otimes A_j^\dagger \cdots \otimes A_j^\dagger$. It is easy to verify that $M_jC_i$ is the tensor product of $k 2^m \times 2^m$ matrices with the $((j-i) \mod k)$-th component being the identity matrix. This is coincident with the position of $d_j$ in the concatenated string when using the encoding matrix $C_i$. Therefore, if the user applies the measurement $M_j$ and later receives the value of $i$, he learns $d_j$ with probability 1, regardless of which $C_i$ the vendor uses. It is however harder to guarantee condition 2 which is implied from the following property:

2’. For any unitary vector $u \in \mathbb{C}^n$ where $n = 2^{km}$,

$$H_2(C_0u) + \cdots + H_2(C_{k-1}u) \geq (k-1) \log n. \quad (8)$$

According to Theorem 2.1 when $k = 2$, this property is satisfied by letting $C_0 = I$ and $C_1 = W_n$. However, we do not know the existence of such matrices for $k > 2$.
Given the difficulty of finding matrices that satisfy the condition \((2')\), we can instead consider the case when the user is honest, i.e. the user always applies one of the measurements \(M_0, \ldots, M_{k-1}\) so to learn one of \(d_i\)'s for sure. In this case, to minimize the information leaked, we would like to minimize \(L_\infty(A_i^\dagger A_i)\) for \(i \neq j\) by Theorem 2.1. For example, if we can construct a set of \(k\) matrices \(A_0, \ldots, A_{k-1}\) such that \(A_i^\dagger A_j\) is Hadamard for any \(i \neq j\), then the scheme using Eq. 7 has the honest user learning no information other than the target item. Such sets of matrices exist for \(k \leq 2m+1\). Specifically, for any \(m \geq 1\), there exist \(2m+1\) complex matrices \(A_i\)'s of dimension \(2^m \times 2^m\) such that \(A_i^\dagger A_j\) is Hadamard for any \(i \neq j\) [4]. We refer to [9] for a simpler construction. Therefore, using the construction in [9] as the encoding matrices, we can achieve perfect privacy for honest users as long as \(k \leq 2m+1\).

Furthermore, by Theorem 2.1, the encoding of Eq. 4 with \(A_i^\dagger A_j\) Hadamard for any \(i \neq j\) leaks at most \(km/2 = \frac{1}{2} \log n\) bits of information even when users pick arbitrary measurements because

\[
H_2(C_0 u) + \cdots + H_2(C_{k-1} u) = \frac{1}{k-1} \sum_{i \neq j} (H_2(C_i u) + H_2(C_j u)) \geq \frac{k}{2} \log n . \quad \text{by Theorem 2.1}
\]

Hence, the encoding leaks at most \(\log n - \frac{1}{2} \sum_i H_2(C_i u) \leq \frac{1}{2} \log n\) bits. The convexity argument of Section 2.4 applies in this case as well. Thus instead of the \(m\) bits an honest user learns, for general measurements we have the weaker bound where the user could learn up to \(km/2\) bits.

While we are unable to show any bound other than Eq. 9 by numerical experiments with small values of \(k\) and \(m\), we observe that the number of bits leaked is approximately \(0.4k^{0.7}m\) bits. We note that the lower bound on the sum of entropies in [26] for \(N+1\) complementary observables in \(N\)-dimensional Hilbert space, does not apply to our case where the dimension of the encoding matrices is \(2^{km}\), much larger than \(k\), the number of matrices. In addition, the construction in [9] is easy to implement physically as they are a multiplication of a diagonal matrix and the Walsh-Hadamard transform.

**Example.** As a concrete example when \(k = 3\), consider three matrices \(A_0, A_1, A_2\) defined as follows. We let \(A_0 = I, A_1 = \alpha_1 \otimes \cdots \otimes \alpha_1, A_2 = \alpha_2 \otimes \cdots \otimes \alpha_2\) where

\[
\alpha_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \alpha_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}
\]

It can be readily verified that \(A_0, A_1, A_2\) satisfy the property that \(A_i^\dagger A_j\) is Hadamard for \(i \neq j\).

The construction in [9] only works when \(k \leq 2^m + 1\). For larger \(k\)'s, we observe that if we simply pick random unitary matrix according to Haar measure [13], then each \(A_i^\dagger A_j\) is nearly Hadamard by the following lemma.
Lemma 3.1 For two randomly picked $\ell \times \ell$ orthonormal matrices $A, B,$

$$\text{Prob}\{L_\infty(A^\dagger B) \geq t\} \leq 4\ell^2 e^{-t^2/2}.$$  

Proof. The proof follows from the measure concentration result on sphere [18]. That is, for any two randomly picked unit vector $u,v$, \[\text{Prob}\{|u \cdot v| \geq t\} \leq 4e^{-t^2 \ell/2}. \]

Thus, if we pick $k$ random $2^m \times 2^m$ orthonormal matrices $A_0, \cdots, A_{k-1}$, then

$$\text{Prob}\{\exists i \neq j \; L_\infty(A_i^\dagger A_j) \geq t\} = O(k^2 2^{2m} e^{-t^2 2^m/2}). \tag{10}$$

If we let $t \geq c \log k + m$ for some constant $c > 0$, then with high probability, $L_\infty(A_i^\dagger A_j) \leq t$ for all $0 \leq i, j \leq k-1$ and $i \neq j$. Thus, we have the following.

Theorem 3.1 For $m = \Omega(\log k)$, if we pick $k$ random orthonormal matrices, then with high probability, for an honest user, the information he learns about the other items is $O(k \log(m + \log k))$ bits.

Proof. By Eq. (10) the information an honest user learns about any other item is $m - \log t^2 = O(\log(m + \log k))$ bits. Thus, in total it is $O(k \log(m + \log k))$ bits.

One problem with random matrices is that it is hard to realize an encoding and perform a measurement. It would be good if each $A_i$ can be further decomposed into the tensor product of smaller matrices. This can be done when $k$ is much smaller than $m$. Let $r = \Omega(2^k m)$. We pick a random $r \times r$ matrix $B_i$, and let $A_i$ be the $m/\log r$ tensor product of $B_i$. It can be shown by a similar argument that an honest user learns $o(m)$ bits of the other items in addition to the item he chooses. In the case when $k$ is much smaller than $m$, $r$ is much smaller than $2^m$, and it reduces the complexity of encoding and measurement.

In the above discussion, we consider the natural choice where the number of encodings used is the same as the number of items. This, however, does not have to be the case. More generally, we can assume that there are $K$ encodings where $K \neq k$. It is possible that by using $K > k$, less information is leaked.

To further reduce the complexity, in the above construction, we can pick a unitary matrix $A$ and let $A_i = A^i$ for $0 \leq i \leq k-1$. This way, the encoding and measurement are simple as there is only one operator to be implemented. To reduce the information leaked, we would like $A$ satisfy the following properties: $A^k = I$ and $A^i$ is a Hadamard (or nearly Hadamard) matrix for any $1 \leq i \leq k-1$. For $k = 3$, we can let $A$ be the tensor product of the following matrix:

$$\frac{e^{i\pi/12}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$
For general $k$'s, again we do not know how to construct such $A$. It would be interesting to know whether, for any $k$ and sufficiently large $n$, there exists an $n \times n$ unitary matrix $A$ so that $A^k = I$ and $A^i$ is Hadamard for $1 \leq i \leq k - 1$.

When the items contain many bits ($m$ is large), another approach is to encrypt the items and use the above protocol with the corresponding decryption keys (which generally use considerably fewer bits than the size of the encrypted items so the quantum mechanism would not have to deal with the large number of bits in each item). In this case, the user picks an item by arranging the measurement to learn the bits of that item’s key. The limit on how much information the user can learn would then make it difficult to learn multiple items, provided the keys are long enough. For example, with the 2-encoding case discussed above, the keys must be long enough that guessing the remaining bits when half are known is still not feasible.

### 4 Discussion

In this paper, we described a simple, private database query protocol using a quantum communication channel. Its ability to maintain privacy for the vendor relies on an assumption of limited coherence times for storing and manipulating quantum states. Thus this protocol is not only suitable for early development of quantum information technology with limited capability, but specifically takes advantage of those limitations.

Because the user can choose to learn about combinations of bits of the database items instead of just a single item for sure, our protocol presents a larger range of choices for the user than conventional treatments of oblivious transfer. The extent to which this would be beneficial depends on the economic context, and associated incentives, in which the protocol is used. One possible application is as a component of digital property rights management. Specifically, the protocol could be useful in situations where the main economic value is from the combined inputs of user and vendor, rather than simply the data from the vendor. That is, private computation of a function of both the vendor’s data and the user’s choice as influenced by private information held by the user.

In this case, with a reasonably large number of items and user choices, even if the user were to reveal the result to other potential users, that information would likely have low value to the other users unless they happened to wish to make the same choice as the original user. Thus those additional users would also need to purchase the information from the vendor rather than attempting to free ride on a single user’s purchase.

An interesting direction for future study is generalizing the protocol to multiple users who have access to additional quantum channels among themselves. In particular, in some economic scenarios, users may wish to ensure coordinated choices while still maintaining as much privacy as possible. In such situations, it would be useful to identify any benefits of the quantum channel among users, particularly if limited to pairwise entanglement which is easier to implement.
than more general entangled states. The potential economic benefits of such a protocol should also be compared with that available with using classical correlation [19]. Moreover, in practice, the prior distribution of database values need not be the uniform distribution we considered and it would be of interest to evaluate the consequences of such prior knowledge on the part of the receiver of the quantum state.

Our quantum mechanism can be simulated classically by having each player to send a choice of operator to a trusted third party. This observation, which also applies to other quantum games [28], means the practical benefit of such quantum mechanisms depends on the context of the game, e.g., the differences in security and communication costs as well as the level of trust assumed for the central institution. For instance, the quantum version allows only a single measurement of the outcome rather than revealing the full database, and hence can provide additional privacy for the vendor. Such privacy can also be achieved via conventional cryptographic methods but with security based on the apparent difficulty of solving certain problems, e.g., factoring, rather than inherent in quantum physics. In addition, there is a large overhead when using cryptographic methods, especially when the number of bits involved is small.

Finally, using game theory to evaluate behavior of economic mechanisms gives at best approximations of real human behavior. In this case, rationality dictates that each individual has a full understanding of the quantum mechanical implications of the measurement operator choices. How well this describes the actual behavior of people involved in quantum games is an interesting direction for future work with laboratory experiments involving human subjects.

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References


