Quantum Energy Levels and Classical Periodic Orbits: Discreteness and Statistical Duality

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It is shown that both universal and non-universal correlations must exist between classical periodic orbits in order that Gutzwiller's semiclassical trace formula is consistent with a real, discrete quantum energy spectrum. Formulae for the two-point correlations are derived. The universal correlations are consistent with those conjectured by Argaman *et al.* (1993). Likewise, both universal and non-universal correlations must exist between quantum energy levels in order that the trace formula be consistent with the fact that periodic orbit actions are real and discrete. In this case, the two-point correlations implied are consistent with random matrix theory and previous semiclassical calculations. These ideas are illustrated with reference to the primes and the Riemann zeros.

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1 Introduction

The trace formula (Gutzwiller 1971) provides a semiclassical link between the quantum energy levels and classical periodic orbits in bound systems. Our purpose here is, as a contribution to Martin Gutzwiller's 75$^{\text{th}}$ birthday celebrations, to use this centrally important result to investigate the consequences for the periodic orbits of the fundamental fact that the quantum energy spectrum is real and discrete in such systems.

We argue that, in the absence of systematic degeneracies, both universal and non-universal correlations must exist between these orbits as a direct result of quantum discreteness. The universal correlations are consistent with those conjectured by Argaman *et al.* (1993). In addition, we exploit the quantum-classical duality embodied in the trace formula to recover certain components of the universal and non-universal correlations in the energy spectrum (Berry 1985; Agam *et al.* 1995; Bogomolny & Keating 1996b), in this case assuming only that the periodic orbit actions are not systematically degenerate.

For simplicity, we consider two-dimensional, classically chaotic systems, although, as we shall indicate, our results generalize straightforwardly. Denoting the quantum energy spectrum by $\{E_n\}$, the density of states is

$$d(E) = \sum_n \delta(E - E_n) = \overline{d}(E) + d_{\text{osc}}(E)\,, \tag{1}$$

where the mean density $\overline{d}$ is given, to leading order semiclassically, by

$$\overline{d}(E) \sim \frac{\Omega(E)}{4\pi^2\hbar^2}\,, \tag{2}$$

$\Omega(E)$ being the volume of the phase-space shell of energy E. In the semiclassical limit, $d_{\text{osc}}(E)$ can be expressed, via Gutzwiller's trace formula, in terms of the periodic orbits of the classical dynamics (Gutzwiller 1971):

$$d_{\text{osc}}(E) \simeq \frac{1}{\pi\hbar} \sum_p A_p(E) \cos\left(\frac{S_p(E)}{\hbar} - \frac{\pi}{2}\alpha_p \right)\,, \tag{3}$$

where p labels the periodic orbits, which have actions $S_p(E)$ and Maslov indices α_p, and

$$A_p(E) = \frac{T_p(E)}{r_p\sqrt{|\det(\mathbf{M}_p - \mathbf{I})|}}\,, \tag{4}$$

$T_p(E) = \mathrm{d}S_p(E)/\mathrm{d}E$ being the period of the p-th orbit, $\mathbf{M}_p$ the monodromy matrix, and r_p the repetition number. The Maslov index plays only a marginal role in our analysis, and so often, for simplicity, we will incorporate it into the amplitude.

We shall be concerned with the statistical distributions of energy levels and periodic orbits. The spectral two-point correlation function, $\tilde{R}_2(x)$, is defined in terms of the fluctuating part of the density of states by

$$\tilde{R}_2(x;E) = \frac{1}{\overline{d}^2} \left\langle d_{osc}\left(E + \frac{x}{2\overline{d}}\right) d_{osc}\left(E - \frac{x}{2\overline{d}}\right)\right\rangle_E , \tag{5}$$

where the angular brackets, $\langle\,.\,\rangle_E$, denote an energy average over a range ΔE that satisfies $\overline{d}^{-1} \ll \Delta E \ll E$ (or, if desired, an average over momentum or $\hbar^{-1}$). The Fourier transform of $\tilde{R}_2$ is the form factor

$$K(\tau;E) = \int_{-\infty}^{\infty} \tilde{R}_2(x;E) e^{2\pi i \tau x}\,\mathrm{d}x\,. \tag{6}$$

Substituting the trace formula (3) into (6) gives a semiclassical representation for the form factor (Berry 1985),

$$K^{sc}(\tau) = \frac{1}{2\pi\hbar\overline{d}} \left\langle \sum_{p,q} A_p A_q \cos\left[\frac{1}{\hbar}\left(S_p - S_q\right)\right] \times \right.$$
$$\left. \times\, \delta\left\{T - \frac{1}{2}\left(T_p + T_q\right)\right\}\right\rangle_E , \tag{7}$$

from which it can be argued that long-time classical ergodicity implies universality (modulo symmetry), consistent with random matrix theory, when $\tau \to 0$ after the semiclassical limit has been taken (Hannay & Ozorio de Almeida 1984; Berry 1985), and that $K(\tau)$ has non-universal structure, related to the short-time dynamics, when $\tau = O(1/\hbar\overline{d})$. It follows from (6) and (5), the fact that the quantum spectrum is real and discrete, and the assumption that it is also not systematically degenerate, that $K(\tau) \to 1$ as $\tau \to \infty$, but demonstrating this semiclassically, directly from (7), is a major unsolved problem.

Our strategy here is to invert this problem. The question we ask is: assuming that $K^{sc}(\tau) \to 1$ as $\tau \to \infty$, what does this imply about the statistical distribution of periodic orbits? This is clearly related to the work of

Argaman *et al.* (1993) and Cohen *et al.* (1998) on action correlations. The differences are, first, that there it was assumed that $K^{sc}(\tau)$ coincides with the appropriate random matrix form for all τ, which is much stronger than the assumptions we make here, and second, that we also recover information about non-universal correlations between the periodic orbits. In addition, we investigate the corresponding universal and nonuniversal correlations between the energy levels that are necessary in order that the trace formula be compatible with the fact that the actions of the periodic orbits are all real and discrete (and assumed non-degenerate). The results to be reported formed the basis of Connors (1998).

Our approach is based on the following observation. For any real, discrete spectrum with no systematic degeneracies, Berry (1985) noted that the Lorentz-smoothed density

$$d_\epsilon(E) = \frac{1}{\pi}\sum_n \frac{\epsilon}{(E-E_n)^2+\epsilon^2} \tag{8}$$

satisfies

$$2\pi\epsilon\, d_\epsilon^2(E) \simeq d_{\frac{\epsilon}{2}}(E) \qquad \text{for} \quad \epsilon\overline{d}(E) \ll 1\,. \tag{9}$$

This will here be referred to as the *quantum distribution rule.* (Corresponding relations can be written down for other choices of smoothing.) It follows that

$$\lim_{\epsilon\to 0} 2\pi\epsilon d_\epsilon^2(E) = d(E)\,. \tag{10}$$

Substituting the trace formula (3) for the denisty of states into (10) and averaging locally with respect to E gives the *semiclassical sum rule* (Berry 1985):

$$\lim_{\epsilon\to 0}\frac{\epsilon}{\pi\hbar^2}\left\langle \sum_{q,p} A_p A_q \cos\left[\frac{1}{\hbar}(S_p - S_q)\right] e^{-\frac{\epsilon}{\hbar}(T_p+T_q)}\right\rangle_E \simeq \overline{d}(E)\,. \tag{11}$$

Essentially, this suggest that the mean level density is encoded into the pairwise distribution of the long periodic orbits satisfying $S_p \simeq S_q$.

Our specific aim here is to determine what the semiclassical sum rule implies about correlations between the periodic orbits. We also investigate the more general semiclassical implications of (10), and the corresponding formulae for correlations in the energy spectrum itself that follow from Fourier-inverting the trace formula. The results are illustrated by reference to the correlations between the Riemann zeros and the primes.

2 Universal periodic orbit correlations

Our goal in this section is to derive information about classical periodic orbit correlations from the semiclassical sum rule (11). To achieve this we remove all quantum variables from (11) by a series of transforms as follows.

The diagonal sum in (11), can be evaluated using the classical sum rule of Hannay & Ozorio de Almeida (1984) to give

$$\sum_p A_p^2 e^{-\frac{2\epsilon}{\hbar}T_p} \simeq g\int_0^\infty T e^{-\frac{2\epsilon}{\hbar}T}\mathrm{d}T = \frac{g\hbar^2}{4\epsilon^2}\,, \tag{12}$$

where g is the mean orbit multiplicity; $g = 1$ for systems without time-reversal invariance and $g = 2$ for systems that are time-reversal invariant. The off-diagonal sum is therefore

$$\left\langle \sum_{\substack{q,p\\ S_p\neq S_q}} A_p A_q \cos\left[\frac{1}{\hbar}(S_p - S_q)\right] e^{-\frac{\epsilon}{\hbar}(T_p+T_q)} \right\rangle_E \simeq \frac{\Omega}{4\pi\epsilon} - \frac{g\hbar^2}{4\epsilon^2}\,, \tag{13}$$

where we have replaced $\overline{d}(E)$ by $\frac{\Omega}{4\pi^2\hbar^2}$, the Thomas-Fermi estimate for two-dimensional systems.

We now take an inverse Laplace transform in $\frac{2\epsilon}{\hbar}$, which introduces a conjugate classical time T. This gives

$$\left\langle \sum_{\substack{q,p\\ S_p\neq S_q}} A_p A_q \cos\left[\frac{1}{\hbar}(S_p - S_q)\right] \delta\left[T - \frac{1}{2}(T_p + T_q)\right] \right\rangle_E \simeq \frac{\Omega}{2\pi\hbar} - gT\,. \tag{14}$$

The condition for the validity of the quantum distribution rule (see (9)), that $\epsilon\overline{d} \sim \epsilon\Omega/\hbar^2 \ll 1$, transforms to $T \gg \frac{\Omega}{\hbar}$.

We next make a further transform in z, defined by

$$z = \frac{1}{\tau} = \frac{2\pi\hbar\overline{d}}{T} = \frac{\Omega}{2\pi T\hbar}\,. \tag{15}$$

Given a function $F(z)$, this transform is defined by

$$\frac{1}{\pi\sqrt{\pi}y}\int_0^\infty\int_{-\infty}^\infty e^{-\frac{x^2}{y^2}}\cos(xz)F(z)\mathrm{d}x\mathrm{d}z = \frac{1}{\pi}\int_0^\infty e^{-\frac{z^2y^2}{4}}F(z)\mathrm{d}z\,. \tag{16}$$

The validity condition now corresponds to $y \gg 1$.

We next define a *classical action correlation function* $C(y,T)$ by

$$C(y,T)=\frac{1}{yT\sqrt{\pi}}\left\langle \sum_{\substack{q,p\\ S_p\neq S_q}} A_pA_q\exp\Big[-\frac{4\pi^2T^2}{\Omega^2y^2}(S_p-S_q)^2\Big]\delta\Big[T-\frac{1}{2}(T_p+T_q)\Big]\right\rangle_E . \tag{17}$$

This correlation function depends solely on properties of the classical dynamics; it picks out pairs of periodic orbits of average period T with action difference $S_p - S_q = O(\Omega^2y^2/T^2)$.

Note that $C(y,T)$ is related to the semiclassical form factor by the transform (16). The large-τ asymptotic behaviour of the form factor thus corresponds to the large y behaviour of $C(y,T)$. It is worth pointing out that this correlation function differs from the one introduced by Argaman *et al.* (1993), the difference being due to the Gaussian transform on the left-hand side of (16).

Returning to the transformed semiclassical sum rule, (14), on applying the transform (16) we get a universal *classical sum rule* for two-dimensional chaotic systems:

$$C(y,T)\simeq\frac{2}{\pi y^2}-\frac{g}{\sqrt{\pi}y}\qquad \text{for } y\gg 1\,, \tag{18}$$

independent of T as $T\to\infty$. This is the main result of this section. It describes correlations between pairs of periodic orbit actions that are necessary for the trace formula to be consistent with a real discrete, non-systematically-degenerate quantum spectrum.

The curious implication of this is that the quantum mechanics of any bound system influences the corresponding classical dynamics; in order that the classical dynamics be semiclassically compatible with quantum mechanics, correlations must exist in the distribution of periodic orbits. In this case we have identified the correlations that are universal in that their behaviour depends only on symmetry.

For a chaotic system with d degrees of freedom ($d\geq 2$) the Thomas-Fermi estimate for the mean density of states is $\overline{d}=\Omega/(2\pi\hbar)^d$, which results in

$$C(y,T)\simeq-\frac{g}{\sqrt{\pi}y}+\frac{1}{\pi y^d}\Big(\frac{T}{\Omega}\Big)^{d-2}\times\begin{cases}\sqrt{\pi}[1.3.5\ldots(d-2)]2^{\frac{d-1}{2}} & \text{if } d \text{ odd}\\ (\frac{d-2}{2})!2^{d-1} & \text{if } d \text{ even}\,,\end{cases} \tag{19}$$

valid for $y^d \gg 1$. This reproduces (18) when $d = 2$.

We note in passing that for integrable systems $C(y,T)$ vanishes in the range under consideration, because the diagonal terms themselves give the large-τ asymptotics of the form factor.

In addition to that identified above, there may be a background component to $C(y,T)$ arising from the uncorrelated component in the distribution of periodic orbits. In cases where the Maslov indices are themselves uncorrelated, each term in the double sum appears with an effectively random phase and hence, in the averaging, terms cancel and there is no background. However, in cases where every Maslov index is zero (modulo 4), the phase factor associated with each term vanishes. The uncorrelated contributions from terms in the double sum in (17) then give rise to a background component, $C_B(y,T)$, which we now compute.

We approximate the amplitude terms using $A_p \simeq T_p \exp(-\frac{1}{2}\lambda T_p)$, with λ the metric entropy. Expressing the actions in terms of their periods $S_p(E) = \sigma(E)T_p(E)$, where $\sigma(E)$ is constant on the energy shell (see Hannay & Ozorio de Almeida 1984), we can rewrite the sums as integrals to give

$$C_B(y,T) = \frac{1}{yT\sqrt{\pi}} \int_{T_p}\int_{T_q} T_p e^{-\frac{1}{2}\lambda T_p} T_q e^{-\frac{1}{2}\overline{\lambda} T_q} \exp\Big[-\frac{4\pi^2 T^2}{\Omega^2 y^2}\sigma^2(E)(T_p - T_q)^2\Big] \times \delta\Big[T - \frac{1}{2}(T_p + T_q)\Big] \frac{e^{h_t T_p}}{T_p} \mathrm{d}T_p \frac{e^{h_t T_q}}{T_q} \mathrm{d}T_q \,. \qquad (20)$$

Assuming that Pesin's theorem applies, that is $\lambda = h_t$, where h_t is the topological entropy,

$$C_B(y,T) = \frac{\Omega}{4\pi\sigma T^2} e^{h_t T} \Big(1 + \mathrm{erf}\Big[\frac{4\pi\sigma T^2}{\Omega y}\Big]\Big) \,. \qquad (21)$$

The analysis leading to (18) is semiclassical and relies on the assumption that the trace formula is able to reproduce a discrete spectrum. The fact that the trace formula is only a semiclassical approximation formally restricts the range of values of y for which $C(y,T)$ can be calculated, because it leads to a divergence in the semiclassical approximation to the form factor (Keating 1994). Specifically, if the divergence occurs when $\frac{1}{z} > \tau^* = O(1)$, then the integral form of $C(y,T)$ is only valid in the range $y < y^* = O(1)$. This is potentially in conflict with the condition that $y \gg 1$ (which corresponds to $\tau \gg 1$) and might mean that our approach is not applicable to typical systems. It suggests that we might have to restrict our interest to cases where the trace formula is exact, such as geodesic motion on surfaces of constant negative curvature and the cat maps.

There are, however, several reasons why the correlations we predict might be more robust than this formal argument suggests. Those systems for which the trace formula is exact are considered to be paradigms of classical chaos, and the fact that universal classical action correlations are predicted in them hints at the existence of similar correlations in other strongly chaotic systems. Put another way, the correlations are a property of the classical dynamics; there is no *a priori* reason why their existence should depend on whether the trace formula is exact or not.

Further evidence is provided by the numerical work of Argaman *et al.*, (1993), Cohen *et al.* (1998), and Tanner (1999), who examined various systems for which the trace formula is a semiclassical approximation and yet found periodic orbit correlations which, at least qualitatively, match the theoretical predictions of Argaman *et al.* (1993).

It is instructive to compare the large-y asymptotic approximation (18) with the formulae that follow from assuming that the form factor is given exactly by one of the corresponding random-matrix expressions. If $K(\tau)$ coincides with the Gaussian Unitary Ensemble (GUE) form factor, then

$$\begin{aligned} C_{GUE}(y) &= \frac{1}{\pi}\int_0^1 \exp\Big(-\frac{y^2z^2}{4}\Big)[z-1]\mathrm{d}z \\ &= \frac{2}{\pi y^2} - \frac{1}{\sqrt{\pi}y} + O\Big(\frac{1}{y^2}e^{-\frac{y^2}{4}}\Big)\,. \end{aligned} \tag{22}$$

Setting $g=1$ in (18), the asymptotic approximation to $C(y,T)$ may be seen to match $C_{GUE}(y)$ up to an exponentially small error. Thus in this case C is almost entirely determind by quantum discreteness.

For systems whose energy-level statistics match those of the Gaussian Orthogonal Ensemble (GOE) of random matrix theory, we find in the same way that

$$C_{GOE}(y) = \frac{2}{\pi y^2} - \frac{2}{\sqrt{\pi}y} + O\Big(\frac{1}{y^4}\Big)\,, \tag{23}$$

which coincides with the $g=2$ form of (18).

The above calculations obviously generalize to higher order semiclassical sum rules. Extending (9) to the cubic case gives

$$d_{\frac{3}{8}\epsilon}(E) \simeq \frac{8}{3}\pi^2\epsilon^2\, d_\epsilon^3(E) \qquad \text{for} \quad \epsilon\overline{d}(E) \ll 1\,. \tag{24}$$

For example, for a two dimensional chaotic system without time-reversal invariance we obtain the following classical correlation formula,

$$C_3(y,T) \simeq \frac{2}{\pi y^2} \qquad \text{for } y \gg 1\,, \tag{25}$$

where $C_3(y,T)$ is defined by

$$C_3(y,T) = \frac{1}{yT^2\sqrt{\pi}}\Bigg\langle \sum_{p_1,p_2,p_3} A_{p_1}A_{p_2}A_{p_3}\exp\Big[-\frac{4\pi^2T^2}{\Omega^2y^2}(S_{p_1}-S_{p_2}-S_{p_3})^2\Big]\times$$
$$\times\delta\Big[T-\frac{1}{2}(T_{p_1}+T_{p_2}+T_{p_3})\Big]\Bigg\rangle_E\,, \tag{26}$$

the sum running over all triples of periodic orbits.

3 Prime correlations

As has been discussed extensively elsewhere (see, for example, Berry & Keating 2000), the Riemann zeta function, $\zeta(s)$, provides a remarkably useful mathematical model for many problems in quantum chaos. Riemann conjectured that all of its complex zeros lie on the line $\mathrm{Re}s = 1/2$. Taking them to be at positions $s = 1/2 + iE_n$, the hights E_n can, conjecturally, be interpreted as quantum energy levels. The density

$$d_R(E) = \sum_n \delta(E - E_n) \tag{27}$$

has an explicit formula in terms of the primes

$$d_R(E) = \overline{d}_R(E) - \frac{1}{\pi}\sum_{p,k}\frac{\log p}{\sqrt{p^k}}\cos(E\log p^k)\,, \tag{28}$$

which plays the role of a trace formula. The primes are thus identified with classical periodic orbits. Here

$$\overline{d}_R(E) \simeq \frac{1}{2\pi}\log(|E|)\,. \tag{29}$$

To illustrate the methods outlined in the previous section, we now analyse prime correlations exactly as we did for periodic orbits. The only difference is due to the fact that the mean density of the Riemann zeros increases as the logarithm of the asymptotic parameter E, rather than as a power of it.

The analogue of the semiclassical sum rule (11) for the Riemann zeros is

$$\lim_{\epsilon\to 0}\frac{\epsilon}{\pi}\Bigg\langle \sum_{\substack{\text{primes}\\ p,q}}\sum_{j,k=1}^{\infty}\frac{\log p\log q}{\sqrt{p^k q^l}}\times$$

$$\times\cos[E(k\log p-l\log q)]e^{-\epsilon(k\log p+l\log q)}\Bigg\rangle_E=\overline{d}_R(E)\,, \tag{30}$$

where the first sum includes all primes p, q.

The diagonal sum may be estimated using the prime number theorem:

$$\sum_{\substack{\text{primes}\\ p}}\frac{\log^2 p}{p}e^{-2\epsilon\log p}\simeq\sum_{n=n_o}^{\infty}\frac{\log n}{n}e^{-2\epsilon\log n}\simeq\int_{\log n_o}^{\infty}xe^{-2\epsilon x}\mathrm{d}x\simeq\frac{1}{4\epsilon^2}\,, \tag{31}$$

to leading order. Higher powers of the primes do not contribute and have been ignored.

We now take an inverse Laplace transform ($2\epsilon\leftrightarrow L$) and apply a transform in $(E\leftrightarrow y)$ as in (16). This gives, for $y\gg e^{-T}$,

$$C_R(y,T)\simeq-\frac{1}{\sqrt{\pi}y}(\log y+T+\frac{\gamma}{2}+\log 2\pi)\,, \tag{32}$$

with $C_R(y,T)$ defined by

$$C_R(y,T)\simeq\frac{1}{\sqrt{\pi}y}\sum_{\substack{\text{primes}\\ p\neq q}}\frac{\log p\log q}{\sqrt{pq}}\exp\left[-\frac{1}{y^2}\log^2\left(\frac{p}{q}\right)\right]\delta\Big[T-\frac{1}{2}\log pq\Big]\,. \tag{33}$$

We have again ignored prime powers because these do not contribute at leading order. As before, the discreteness of the Riemann zeros determines the asymptotic (in y) behaviour of correlations in the pairwise distribution of the primes.

There is also a background component to $C_R(y,T)$ that arises from the uncorrelated component in the distribution of the primes. From the above definition of $C_R(y,T)$, (33), we evaluate the sums independently using the prime number theorem. The background component $C_R^B(y,T)$ is then found to be

$$C_R^B(y,T) \simeq e^T \qquad \text{as} \quad \frac{T}{y} \to \infty \,. \tag{34}$$

Unlike the general case, here we can calculate the off-diagonal double sum explicitly, and hence directly verify this prime correlation formula (32). Whereas in the general case the only means of evaluating the double sum was to use random-matrix results, here we can use the prime correlation information contained in the conjecture of Hardy & Littlewood (1923) to calculate the value of the sums and hence $C_R(y,T)$. This method of estimating prime sums is exactly that previously used to verify the sum rule (30) (Keating 1991), and in calculating other statistical properties of the Riemann zeros (Keating 1993; Bogomolny & Keating 1995, 1996a). We find (Connors 1998) that

$$C_R(y,T) \simeq e^T - \frac{1}{\sqrt{\pi} y}\Big(\log y + T + O(1)\Big)\,, \tag{35}$$

in agreement with the prime correlations, (32), derived from the discreteness of the Riemann zeros and the background component, (34), of $C_R(y,T)$.

4 Correlations and duality

We now return to the quantum distribution rule (9) in an attempt to uncover more information about classical correlations; in particular we will show that the discreteness of the quantum spectrum implies the existence of non-universal classical correlations, in addition to the universal ones already discussed.

For simplicity, we consider the representative case of two-dimensional chaotic billiards which are not time-reversal invariant. The results generalize easily to the time-reversal invariant case and to higher dimensional systems.

4.1 Quantum correlation functions

The density of momentum states and its semiclassical representation are given by

$$d(k) = \sum_{n=-\infty}^{\infty} \delta(k-k_n) = \overline{d}(k) + d_{osc}(k) \tag{36}$$

$$\simeq \frac{1}{2\pi}\Omega k + \frac{1}{\pi}\sum_p A_p \cos(kL_p)\,, \tag{37}$$

where the area of the billiard is Ω, and L_p denotes the periodic orbit lengths.

From (5), a semiclassical expression for $\tilde{R}_2$ is therefore

$$\tilde{R}_2(x) \simeq \frac{1}{2\pi^2}\left\langle \sum_{p,q} A_p A_q \cos\left[(L_p - L_q)k + xL_p\right]\right\rangle_k\,, \tag{38}$$

and for the form factor,

$$K(\tau) \simeq \frac{1}{2\pi\overline{d}}\left\langle \sum_{p,q} A_p A_q \cos[(L_p - L_q)k]\,\delta[L - L_p]\right\rangle_k\,, \tag{39}$$

with $\tau = \frac{L}{2\pi\overline{d}}$.

4.2 Classical correlation functions

We now define

$$d^c(L) = \sum_p A_p \delta(L - L_p) = \overline{d^c}(L) + d^c_{osc}(L)\,. \tag{40}$$

For simplicity, we assume that all Maslov phases (which are here implicit in the amplitudes) vanish and that all Lyapunov exponents are equal (to λ); that is

$$A_p = \frac{L_p}{2\sinh\left(\frac{\lambda L_p}{2}\right)}\,. \tag{41}$$

Then

$$d^c(L) = \frac{L}{2\sinh\left(\frac{\lambda L}{2}\right)}\sum_p \delta(L - L_p)\,, \tag{42}$$

where, in the limit $L \to \infty$, the leading-order asymptotic behaviour of the smooth mean density of states, $\overline{d^c}$, is given by

$$\overline{d^c}(L) \simeq e^{\frac{1}{2}\lambda L} \,. \tag{43}$$

(We note that while the conditions imposed may appear somewhat restrictive, they are satisfied for geodesic motion on surfaces of constant negative curvature and for the cat maps.)

Fourier transforming the trace formula (37), as in Colin de Verdiere (1973) and Chazarain (1974), we have, in the limit $L \to \infty$, that

$$d^c(L) \simeq e^{\frac{1}{2}\lambda L} + 2\sum_{n=1}^{\infty} \cos(k_n L) \,. \tag{44}$$

Defining a classical two-point correlation function by

$$\tilde{R_2}^c(x) = \left\langle d^c_{osc}(L+x) d^c_{osc}(L) \right\rangle_L \,, \tag{45}$$

where the average is over a range of size ΔL around L,

$$\tilde{R_2}^c(x) = \sum_{p,q} A_p A_q \delta(L_p - L_q - x)\delta_{\Delta L}\Big[L - \frac{1}{2}(L_p + L_q)\Big] - e^{\lambda L} \,, \tag{46}$$

where $\delta_{\Delta L}(\alpha)$ is a δ-function of width ΔL centred at L. This classical two-point correlation function is related to $C(y,T)$ by the transform

$$C(y,L) = \frac{1}{yL\sqrt{\pi}} \left\langle \int_{-\infty}^{\infty} \exp\Big(-\frac{x^2}{y^2}\Big)\Big[\tilde{R}^c_{\rm off}(x) + e^L\Big] \mathrm{d}x \right\rangle , \tag{47}$$

where $\tilde{R}^c_{\rm off}$ comprises the off-diagonal terms of $\tilde{R_2}^c(x)$.

Substituting the trace formula (44) for $d^c(L)$ into (46),

$$\tilde{R_2}^c(x) \simeq 2 \left\langle \sum_{m,n} \cos[(k_m - k_n)L + x k_n] \right\rangle_L \,; \tag{48}$$

that is, the classical periodic orbit correlation function can be expressed semiclassically in terms of the quantum eigenmomenta. The diagonal terms in this expression may be summed to give, for systems without time-reversal invariance,

$$\left\langle \tilde{R}_2^c(x) \right\rangle_x = -\frac{\Omega}{\pi x^2}\,, \tag{49}$$

where the average in x is over a range $\Delta x \gg e^{-\frac{1}{2}\lambda L}$. Note that if x is measured in units of $\sqrt{\Omega}$, then this result becomes universal (i.e. system-independent).

We can obviously define a classical form factor by

$$K^c(\xi) = \frac{1}{e^{\lambda L}} \int_{-\infty}^{\infty} e^{2\pi i \xi x} \tilde{R}_2^c\left(\frac{x}{e^{\frac{1}{2}\lambda L}}\right) \mathrm{d}x\,. \tag{50}$$

Taking

$$k = 2\pi e^{\frac{1}{2}\lambda L} \xi\,, \tag{51}$$

we then have that

$$K^c\left(\frac{k}{2\pi e^{\frac{1}{2}\lambda L}}\right) = e^{-\frac{1}{2}\lambda L} \sum_{p,q} A_p A_q e^{ik(L_p - L_q)} \delta_{\Delta L}\left[L - \frac{1}{2}(L_p + L_q)\right] - 2\pi e^{\frac{1}{2}\lambda L}\delta(k)\,. \tag{52}$$

Note that under any ξ-averaging,

$$\left\langle K^c(\xi) \right\rangle_\xi \sim L e^{-\frac{1}{2}\lambda L} \tag{53}$$

as $\xi \to \infty$, with ξ defined in (51).

A semiclassical expression for the classical form factor follows immediately from (48) and (50). In the limit $L \to \infty$,

$$K^c\left(\frac{k}{2\pi e^{\frac{1}{2}\lambda L}}\right) \simeq 2\pi e^{-\frac{1}{2}\lambda L} \left\langle \sum_{n,m} \cos[(k_m - k_n)L]\,\delta[k - k_n] \right\rangle_L\,. \tag{54}$$

In direct analogy with the quantum form factor, here we find δ-functions positioned at the smallest quantum eigenmomenta, on the scale $\xi \simeq k_0 e^{-\frac{1}{2}\lambda L}$, where k_0 is the smallest eigenmomentum. These correspond to non-universal correlations in the distribution of classical periodic orbit lengths, here expressed in terms of the low lying quantum levels. Later we will show that these non-universal classical correlations can also be expressed in terms of the short periodic orbits.

The diagonal approximation for the classical form factor gives

$$\left\langle K^c(\xi) \right\rangle_k = 2\pi e^{-\frac{1}{2}\lambda L} \overline{d}(k)\,, \tag{55}$$

with ξ as above in (51). Here the averaging is over a sufficiently large range that all the off-diagonal terms are removed from the double sum.

4.3 The semiclassical sum rule

The semiclassical sum rule for the quantum density of states, (11), can be written in terms of the quantum form factor as

$$K(\tau) \simeq 1 \qquad \text{when} \qquad \tau \gg 1\,. \tag{56}$$

It can also be written in terms of the classical form factor, (52):

$$\big\langle K^c(\xi)\big\rangle_k \simeq 2\pi e^{-\frac{1}{2}\lambda L}\overline{d}(k)\,, \tag{57}$$

valid in the region $\overline{d} \ll L$ or equivalently, $\xi \simeq ke^{-\frac{1}{2}\lambda L} \ll \sqrt{k}$.

We have already seen that the fact that the quantum spectrum is real and discrete (and not systematically degenerate) is equivalent to the asymptotic behaviour of the quantum form factor. What (57) demonstrates is that it is also equivalent to the diagonal terms of the classical form factor.

4.4 A classical sum rule

We now examine the dual of the semiclassical sum rule for quantum spectra: a *classical sum rule* for the density of periodic orbit lengths.

A classical periodic orbit length spectrum with no systematic degeneracies, which is accurately reproduced by its trace formula (44), satisfies the quantum distribution rule (9). For strongly hyperbolic billiard systems we can remove the weightings of the classical spectrum as a smooth function of the length. We then have that

$$\left\langle \frac{e^{\frac{1}{2}\lambda L}}{L} d^c(L)\right\rangle_L = \lim_{\epsilon\to 0} 2\pi\epsilon \left\langle \left[\frac{e^{\frac{1}{2}\lambda L}}{L} d^c_\epsilon(L)\right]^2\right\rangle_L . \tag{58}$$

Writing this in terms of the trace formula for the classical spectrum gives

$$\frac{4\pi\epsilon}{L}\left\langle \sum_{m,n} \cos[(k_m-k_n)L]e^{-\epsilon(k_m+k_n)}\right\rangle_L \simeq 1\,, \tag{59}$$

with the condition that $\epsilon e^{\frac{1}{2}\lambda L} \ll 1$ to ensure that the smoothed δ-functions do not overlap.

Taking an inverse Laplace transform in the length smoothing parameter $(2\epsilon \leftrightarrow k)$ gives

$$\left\langle \sum_{m,n} \cos[(k_m-k_n)L]\delta\Big[k-\frac{1}{2}(k_m+k_n)\Big]\right\rangle_L \simeq \frac{L}{2\pi}\,, \tag{60}$$

which holds in the semiclassical limit $k \to \infty$. The condition $\epsilon e^{\frac{1}{2}\lambda L} \ll 1$ transforms to $e^{\frac{1}{2}\lambda L} \ll k$.

Writing this in terms of the classical form factor gives

$$K^c(\xi) \simeq L e^{-\frac{1}{2}\lambda L}\,, \tag{61}$$

for $\xi \gg 1$ and with ξ defined by (51). This is the large argument behaviour of $K^c(\xi)$ noted earlier, (53). The classical sum rule therefore corresponds to the asymptotic behaviour of the classical form factor, just as the semiclassical sum rule can be written in terms of the asymptotic behaviour of the quantum form factor.

In terms of the quantum form factor, the classical sum rule for systems without time-reversal invariance is simply

$$K(\tau) \simeq \tau\,, \tag{62}$$

for $\tau \ll \log k/(\lambda \Omega k))$. This is exactly the diagonal approximation to the quantum form factor which is the correct (GUE) behaviour of $K(\tau)$ for small τ.

We thus see that the classical sum rule, the dual of the semiclassical sum rule for the quantum spectrum, corresponds to the diagonal approximation to the quantum form factor, and the large-ξ asymptotics of the classical form factor.

4.5 A generalized semiclassical sum rule

We can now extend this dualistic approach and derive a generalized semiclassical sum rule (GSCSR) for both the quantum density of states and, in the next section, the classical density $d^c(L)$.

As in Keating (1991), we introduce a phase factor, $\cos(kx)$, into the semiclassical sum rule. In the limit $\epsilon \to 0$, we therefore have, from (11), that

$$\begin{aligned}
&\frac{\epsilon}{2\pi}\left\langle \sum_{p,q} A_p A_q \cos[(L_p - L_q \pm x)k] e^{-\epsilon(L_p+L_q)} \right\rangle_k \\
&\quad \simeq \left\langle \overline{d}(k)\cos(kx)\right\rangle_k - \frac{1}{2\pi}\left\langle \sum A_p \cos[(L_p - x)k] e^{-\frac{1}{2}\epsilon L_p} \right\rangle_k .
\end{aligned} \tag{63}$$

with the condition that $\epsilon \overline{d}(k) \ll 1$.

We now follow the method already used for the semiclassical sum rule and take an inverse Laplace transform of (63) in the momentum smoothing parameter ($2\epsilon \leftrightarrow L$). This gives

$$\frac{1}{4\pi}\left\langle \sum_{p,q} A_p A_q \cos[(L_p - L_q \pm x)k]\delta\Big[L - \frac{1}{2}(L_p + L_q)\Big] \right\rangle_k$$
$$\simeq \langle \overline{d}(k)\cos(kx)\rangle_k - \frac{1}{2\pi}\left\langle \sum A_p \cos[(L_p - x)k]\Theta\Big[L - \frac{1}{4}L_p\Big] \right\rangle_k , \quad (64)$$

in the limit $L \to \infty$, for $k \ll L$.

We next make the following transformation, defined by its action on a function $f(k)$ by

$$\frac{4y}{\pi}\int_0^\infty \frac{1}{y^2 + x^2}\int_0^\infty \cos(kx) f(k) \mathrm{d}x\mathrm{d}k = 2\int_0^\infty e^{-ky} f(k)\mathrm{d}k\,. \quad (65)$$

Implementing this k-integral constitutes an evaluation of the k-averaging denoted by the angular brackets. We arrive at the result

$$\frac{1}{4}\sum_{p,q} A_p A_q\, \delta_y[L_p - L_q \pm x]\delta\Big[L - \frac{1}{2}(L_p + L_q)\Big]$$
$$\simeq \frac{\Omega}{2\pi}\frac{y^2 - x^2}{(y^2 + x^2)^2} - \frac{1}{2}\sum_p \delta_y[L_p - x]\Theta\Big[L - \frac{1}{4}L_p\Big] , \quad (66)$$

where $\delta_y[\beta] = y/[\pi(y^2 + \beta^2)]$, a Lorentzian of width y centred at $\beta = 0$ which gives a δ-function in the limit $y \to 0$. The condition $k \ll L$ transforms to $y \gg \frac{1}{L}$. Taking the limit $y \to 0$ therefore requires $L \to \infty$. Rewriting this in terms of the two-point classical correlation function, (48), we have that

$$\tilde{R}_2^c(x) \simeq -\frac{\Omega}{\pi x^2} - \lim_{y\to 0}\sum_{L_p < 4L} \delta_y[L_p - x]\,. \quad (67)$$

The first term is exactly the diagonal approximation to the classical two-point correlation function (49). This is the leading order behaviour of the universal correlations which exist in the distribution of classical periodic orbits. There are oscillatory corrections to this which appear in the (unsmoothed) result of Argaman *et al.* (1993) for systems with RMT spectral statistics.

The GSCSR for the quantum density of states therefore implies that there exist non-universal correlations in the distribution of long classical periodic orbits that can be described in terms of the short periodic orbits. It is thus an explicit example of *bootstrapping*. Note that our derivation of these system-specific δ-functions requires no assumptions regarding the statistical distribution of high-lying quantum energy levels; they are correlations arising from the fact that the quantum spectrum is discrete, non-degenerate and is described by a semiclassical trace formula.

4.6 A generalized classical sum rule

Introducing into (59) a phase factor, $\cos(kx)$, in the same way as in the previous section, allows us to write down a *generalized classical sum rule*:

$$2\pi\epsilon\Big\langle \sum_{m,n}\cos[(k_m-k_n\pm x)L]e^{-\epsilon(k_m+k_n)}\Big\rangle_L \simeq \big\langle L\cos(xL)\big\rangle_L + Le^{-\frac{1}{2}\lambda L}\Big\langle\sum_n \cos[(k_n-x)L]e^{-\frac{1}{2}\epsilon k_n}\Big\rangle_L, \qquad (68)$$

with the condition $\epsilon e^{\frac{1}{2}\lambda L}\ll 1$.

As for the quantum case, we now implement an inverse Laplace transform ($2\epsilon\leftrightarrow k$), giving

$$\pi\Big\langle\sum_{m,n}\Big(\cos[(k_m-k_n\pm x)L]\Big)\delta\Big[k-\frac{1}{2}(k_m+k_n)\Big]\Big\rangle_L \simeq \big\langle L\cos(xL)\big\rangle_L + Le^{-\frac{1}{2}\lambda L}\Big\langle\sum_n\cos[(k_n-x)L]\Theta\Big[k-\frac{1}{4}k_n\Big]\Big\rangle_L \qquad (69)$$

in the limit $k\to\infty$. The condition $\epsilon e^{\frac{1}{2}\lambda L}\ll 1$ transforms to $e^{\frac{1}{2}\lambda L}\ll k$.

Laplace transforming ($L\leftrightarrow y$) (69) gives

$$2\pi^2\sum_{m,n}\delta_y[k_m-k_n-x]\,\delta\Big[k-\frac{1}{2}(k_m+k_n)\Big] \simeq \frac{y^2-x^2}{[y^2+x^2]^2}+\sum_n\frac{\Big(y+\frac{\lambda}{2}\Big)^2-(k_n-x)^2}{\Big[\Big(y+\frac{\lambda}{2}\Big)^2+(k_n-x)^2\Big]^2}\Theta\Big[k-\frac{1}{4}k_n\Big]. \qquad (70)$$

In the limit $y \to 0$, we can express this formula in terms of the quantum two-point correlation function:

$$\tilde{R}_2(x) = -\frac{1}{2\pi^2 x^2} + \frac{1}{2\pi^2} \sum_n \frac{\left(\frac{\lambda}{2}\right)^2 - (k_n - x)^2}{\left[\left(\frac{\lambda}{2}\right)^2 + (k_n - x)^2\right]^2} - \overline{d}^2 . \qquad (71)$$

The sum includes only those quantum eigenmomenta satisfying $k_n < 4k$, but since the sum converges we allow all terms to contribute.

The first term on the right-hand side is the non-oscillatory component of the GUE two-point correlation function. The oscillatory component is not reproduced here. We initially made an ϵ-smoothing of the spectrum in order to remove the off-diagonal terms in the squared spectrum of the quantum distribution rule (9). We therefore only recover the diagonal behaviour of $\tilde{R}_2(x)$; the oscillatory component arises from the off-diagonal terms, as shown by Bogomolny & Keating (1996b).

In addition to the universal component of $\tilde{R}_2(x)$ arising from the diagonal approximation, we have here recovered non-universal two-point correlations in the quantum density of states. These have been derived from the fact that the classical spectrum is discrete with no assumptions regarding the distribution of classical periodic orbit lengths, or quantum energy levels. We have imposed conditions on the classical dynamics of the system regarding the Lyapunov exponents and the Maslov indices, but since the resulting formula is entirely quantum we expect this sophisticated form of quantum bootstrapping, between the two-point statistics of the high energy levels and the low-lying energy levels, to be independent of detailed assumptions about the classical dynamics and the methods of semiclassical analysis.

These non-universal correlations are exactly those related to the short periodic orbits of the system, as seen in the semiclassical formula for the quantum form factor by Berry (1985). Bogomolny & Keating (1996b) and Agam *et al.* (1995) showed that they can be expressed in terms of the analytic structure of the Ruelle zeta function. Under the assumptions made at the beginning of this section, that the Lyapunov exponents are all equal and all Maslov indices are zero mod 4, the (classical) Ruelle zeta function can be considered as an analytic continuation of the Selberg zeta function, which is related to the spectral determinant of the quantum Hamiltonian. Bogomolny & Keating (1996b) not only uncovered the same universal and non-universal correlations for the (diagonal part of) $\tilde{R}_2(x)$, but also summed the off-diagonal terms to recover the oscillatory part of the GUE two-point correlation function. The appropriate part of their results, written in terms of the Ruelle zeta function, can be expressed as in (71). We make this connection explicitly for the Riemann case below. This formula for $\tilde{R}_2(x)$ is therefore a strong confirmation of the integrity of our method.

5 The Riemann zeta function

The calculations of the previous sections can be worked through for the case of the Riemann zeta function. In this particular example, the behaviour of the two-point correlation function, $\tilde{R}_2(x)$, implied by the generalized prime sum rule coincides with the results of Bogomolny & Keating (1996b).

5.1 The primes

The density of the logarithms of the primes is defined (by analogy with the classical density of states in Section 4.2) to be

$$d^p(L) = \sum_p \sum_k \frac{\log p}{\sqrt{p^k}} \delta(L - k \log p) . \tag{72}$$

This can be expressed (Banham 1995) in terms of the Riemann zeros as

$$d^p(L) \simeq e^{\frac{1}{2}L} - 2 \sum_{n=1}^{\infty} \cos(E_n L) . \tag{73}$$

We define a two-point correlation function on the prime spectrum to be

$$\tilde{R}_2{}^p(x) = \sum_{p^k, q^l} \frac{\log p \log q}{\sqrt{p^k q^l}} \delta(k \log p - l \log q - x) \delta_{\Delta L}\Big[L - \frac{1}{2} \log p^k q^l\Big] - e^L , \tag{74}$$

where $\delta_{\Delta L}(\alpha)$ is a ΔL-width δ-function with centre L. This can be written in terms of the Riemann zeros, from (73), as

$$\tilde{R}_2{}^p(x) = 2 \Big\langle \sum_{m,n} \cos[(E_m - E_n)L + xE_n] \Big\rangle_L . \tag{75}$$

Defining $\xi = \frac{E}{2\pi e^{\frac{1}{2}L}}$, the prime form factor is

$$K^p(\xi) = e^{-\frac{1}{2}L} \sum_{p,q} \frac{\log p \log q}{\sqrt{p^k q^l}} e^{iE \log \frac{p}{q}} \delta_{\Delta L}\Big[L - \frac{1}{2}\log pq\Big] - 2\pi e^{\frac{1}{2}L}\delta(E)\,. \quad (76)$$

The asymptotic behaviour of $K^p(\xi)$ is easily determined. In the limit $\xi \to \infty$, any finite-width averaging removes all off-diagonal terms, and we find that

$$\big\langle K^p(\xi)\big\rangle_\xi \simeq L e^{-\frac{1}{2}L} \qquad \text{as} \quad L \to \infty\,. \quad (77)$$

Semiclassically, the prime form factor can be expressed in terms of the Riemann zeros as

$$K^p(\xi) = 2\pi e^{-\frac{1}{2}L} \Big\langle \sum_{n,m} \cos[(E_m - E_n)L]\, \delta[E - E_n] \Big\rangle_L\,. \quad (78)$$

Notice that δ-functions occur at the low-lying Riemann zeros on the scale $\xi \simeq E_0 e^{-\frac{1}{2}L}$, where E_0 is the Riemann zero with smallest imaginary part. These δ-functions describe non-universal correlations in the distribution of prime numbers related to the low-lying Riemann zeros (rather than the universal correlations described by the Hardy & Littlewood conjectures). This non-universality is the analogue of that seen earlier, in the distribution of periodic orbits.

The diagonal approximation to the prime form factor gives

$$\big\langle K^p(\xi)\big\rangle_E = 2\pi e^{-\frac{1}{2}L} \overline{d}_R(E)\,. \quad (79)$$

valid in the limit as $\xi \to 0$.

5.2 The Riemann zeta sum rule

Using the prime conjectures of Hardy & Littlewood (1923), the sum rule for the Riemann zeta function, (30), was explicitly verified by Keating (1991), who evaluated the double sum over primes and recovered $\overline{d}_R$.

In terms of the form factor this semiclassical sum rule states that

$$K(\tau) \simeq 1\,, \quad (80)$$

in the region $\tau \gg 1$.

In terms of the prime form factor this can be written

$$\left\langle K^p(\xi)\right\rangle_\xi \simeq 2\pi\overline{d}_R e^{-\frac{1}{2}L} \tag{81}$$

for $\overline{d}_R \ll L$ or equivalently, $\xi \simeq Ee^{-\frac{1}{2}L} \ll \sqrt{E}$. This is the diagonal approximation to $K^p(\xi)$.

The semiclassical sum rule therefore corresponds to the diagonal approximation, or the averaged behaviour, of the prime form factor, and to the asymptotic behaviour of the form factor of the Riemann zeros.

5.3 A prime sum rule

The primes can be treated in the same way to give a formula analogous to the classical sum rule. After removing the degeneracy weightings, in the limit $\epsilon \to 0$,

$$\frac{4\pi\epsilon}{L}\left\langle \sum_{m,n} \cos[(E_m - E_n)L]e^{-\epsilon(E_m+E_n)}\right\rangle_L \simeq 1\,. \tag{82}$$

In terms of the prime form factor this states that

$$K^p(\xi) \simeq Le^{-\frac{1}{2}L} \qquad \text{when} \quad \xi \gg 1\,, \tag{83}$$

and in terms of the form factor, it can be written

$$K(\tau) \simeq \tau \qquad \text{when} \quad \tau \ll 1\,. \tag{84}$$

This is the diagonal, or averaged behaviour of the form factor in this region.

The discreteness of the prime spectrum therefore corresponds to the diagonal approximation to the form factor of the Riemann zeros, and to the asymptotic behaviour of the prime form factor.

5.4 The generalized Riemann sum rule

Rewriting the generalized semiclassical sum rule, (63), for the Riemann case gives, in the limit $\epsilon \to 0$,

$$\begin{aligned}
&\frac{\epsilon}{2\pi}\left\langle \sum_{p,q} \frac{\log p \log q}{\sqrt{pq}}\left(\cos\left[E\log\frac{p}{\beta q}\right] + \cos\left[E\log\frac{\beta p}{q}\right]\right)e^{-\epsilon\log pq}\right\rangle_E \\
&\quad \simeq \left\langle \overline{d}_R \cos(E\log\beta)\right\rangle_E - \frac{1}{2\pi}\left\langle \sum \frac{\log p}{\sqrt{p^k}}\cos\left[E\log\frac{p^k}{\beta}\right]e^{-\frac{1}{2}\epsilon\log p^k}\right\rangle_E .
\end{aligned} \tag{85}$$

Applying the now familiar sequence of transforms and expressing the result in terms of the two-point prime correlation function (74), we have that

$$\tilde{R}_2^p(x) \simeq -\frac{1}{2x} - \lim_{y\to 0} \sum_{p^k<e^{4L}} \frac{\log p}{\sqrt{p^k}}\, \delta_y[x - \log p^k] - e^L\,. \tag{86}$$

for $y \ll e^L$, with $\delta_y[x] = y/[\pi(y^2+x^2)]$.

Connors & Keating (2001) use the prime conjectures of Hardy & Littlewood (1923) to evaluate the double sum (inherent in $\tilde{R}_2^p(x)$) explicitly, recovering the delicate structure of Kronecker δ-functions positioned at prime powers, $\beta = p^k$.

5.5 A generalized prime sum rule

Introducing a phase factor, $\cos(xL)$, into the prime sum rule (82) gives, in the limit $\epsilon \to 0$,

$$\begin{aligned} 2\pi\epsilon\frac{e^{\frac{1}{2}L}}{L}\Big\langle \sum_{m,n} \cos[(E_m - E_n \pm x)L] e^{-\epsilon(E_m+E_n)} \Big\rangle_L \\ \simeq \big\langle e^{\frac{1}{2}L}\cos(xL)\big\rangle_L - \Big\langle \sum_n \cos[(E_n - x)L]e^{-\frac{1}{2}\epsilon E_n}\Big\rangle_L\,. \end{aligned} \tag{87}$$

The usual sequence of transforms begins by taking an inverse Laplace transform ($2\epsilon \leftrightarrow E$). The Fourier transform (in L) that follows, requires a convolution to maintain the condition $L \ll \log E$. This results in

$$\begin{aligned} 2\pi \sum_{m,n} \frac{y}{y^2 + (E_m - E_n \pm x)^2}\,\delta\Big[E - \frac{1}{2}(E_m+E_n)\Big] \\ \simeq 2\frac{y^2-x^2}{[y^2+x^2]^2} - 2\sum_n \frac{\left(y+\frac{1}{2}\right)^2 - (E_n - x)^2}{\left[\left(y+\frac{1}{2}\right)^2 + (E_n-x)^2\right]^2}\Theta(E - \frac{1}{4}E_n)\,. \end{aligned} \tag{88}$$

In terms of the two-point correlation function, in the limit $y \to 0$ this is

$$\tilde{R}_2(x) \simeq -\frac{1}{2\pi^2 x^2} - \frac{1}{2\pi^2}\sum_{E_n<4E} \frac{\frac{1}{4} - (E_n - x)^2}{\left[\frac{1}{4} + (E_n - x)^2\right]^2} - \overline{d}^2\,. \tag{89}$$

This formula is equivalent to a result obtained by Bogomolny & Keating (1996b) who express the correlations of $\tilde{R}_2(x)$ in terms of the behaviour of the Riemann zeta function near its pole. It derives from the discreteness of the prime spectrum and describes both universal (GUE) and non-universal (non-GUE) correlations in the distribution of Riemann zeros. In addition to the leading order GUE behaviour that arises from the prime sum rule, here we see the existence of (finite width) Lorentzians centred at each Riemann zero. These represent fluctuations in the distribution of Riemann zeros that are not predicted by RMT. They are the same non-universal features described in terms of primes by Berry (1988) and, as already noted, in terms of the analytic structure of $\zeta(1+ix)$ by Bogomolny & Keating (1996b).

As in the billiard case, we only recover the non-oscillatory part of the GUE two-point correlation function. The smoothing required to construct the quantum distribution rule removes the off-diagonal terms and it is these which reproduce the oscillatory terms.

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