A Non-Markovian version of Pitman’s $2M - X$ Theorem

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Pitman's representation theorem, telegrapher's process

Let $(\xi_k, k \geq 0)$ be a Markov chain on $\{-1, +1\}$ with $\xi_0 = 1$ and transition probabilities $P(\xi_{k+1} = 1 | \xi_k = 1) = a$ and $P(\xi_{k+1} = -1 | \xi_k = -1) = b < a$. Set $X_0 = 0, X_n = \xi_1 + \cdots + \xi_n$ and $M_n = \max_{0 \leq k \leq n} X_k$. We prove that the process $2M - X$ has the same law as that of $X$ conditioned to stay non-negative.

Pitman’s representation theorem [18] states that, if $(X_t, t \geq 0)$ is a standard Brownian motion and $M_t = \max_{s \leq t} X_s$, then $2M - X$ has the same law as the 3-dimensional Bessel process. This was extended in [19] to the case of non-zero drift, where it is shown that, if $X_t$ is a standard Brownian motion with drift, then $2M - X$ is a certain diffusion process. This diffusion has the significant property that it can be interpreted as the law of $X$ conditioned to stay positive (in an appropriate sense). Pitman’s theorem has the following discrete analogue [18, 15]: if $X$ is a simple random walk with non-negative drift (in continuous or discrete time) then $2M - X$ has the same law as $X$ conditioned to stay non-negative (for the symmetric random walk this conditioning is in the sense of Doob). Here we present a non-Markovian version of Pitman’s theorem. Let $(\xi_k, k \geq 0)$ be a Markov chain on $\{-1, +1\}$ with $\xi_0 = 1$ and transition probabilities $P(\xi_{k+1} = 1 | \xi_k = 1) = a$ and $P(\xi_{k+1} = -1 | \xi_k = -1) = b$. We will assume that $1 > a > b > 0$. Set $X_0 = 0, X_n = \xi_1 + \cdots + \xi_n$ and $M_n = \max_{0 \leq k \leq n} X_k$. 

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A non-Markovian version of Pitman's $2M - X$ theorem

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Abstract

Let $(\xi_k, \ k \geq 0)$ be a Markov chain on $\{-1, +1\}$ with $\xi_0 = 1$ and transition probabilities $P(\xi_{k+1} = 1 \mid \xi_k = 1) = a$ and $P(\xi_{k+1} = -1 \mid \xi_k = -1) = b < a$. Set $X_0 = 0$, $X_n = \xi_1 + \cdots + \xi_n$ and $M_n = \max_{0 \leq k \leq n} X_k$. We prove that the process $2M - X$ has the same law as that of $X$ conditioned to stay non-negative.

Pitman’s representation theorem [18] states that, if $(X_t, \ t \geq 0)$ is a standard Brownian motion and $M_t = \max_{s \leq t} X_s$, then $2M - X$ has the same law as the 3-dimensional Bessel process. This was extended in [19] to the case of non-zero drift, where it is shown that, if $X_t$ is a standard Brownian motion with drift, then $2M - X$ is a certain diffusion process. This diffusion has the significant property that it can be interpreted as the law of $X$ conditioned to stay positive (in an appropriate sense). Pitman’s theorem has the following discrete analogue [18, 15]: if $X$ is a simple random walk with non-negative drift (in continuous or discrete time) then $2M - X$ has the same law as $X$ conditioned to stay non-negative (for the symmetric random walk this conditioning is in the sense of Doob).

Here we present a non-Markovian version of Pitman’s theorem. Let $(\xi_k, \ k \geq 0)$ be a Markov chain on $\{-1, +1\}$ with $\xi_0 = 1$ and transition probabilities $P(\xi_{k+1} = 1 \mid \xi_k = 1) = a$ and $P(\xi_{k+1} = -1 \mid \xi_k = -1) = b$. We will assume that $1 > a > b > 0$. Set $X_0 = 0$, $X_n = \xi_1 + \cdots + \xi_n$ and $M_n = \max_{0 \leq k \leq n} X_k$.

Theorem 1 The process $2M - X$ has the same law as that of $X$ conditioned to stay non-negative.

Note that, if $b = 1 - a$, then $X$ is a simple random walk with drift and we recover the original statement of Pitman’s theorem in discrete time.

To prove Theorem 1, we first consider a two-sided stationary version of $\xi$, which we denote by $(\eta_k, \ k \in \mathbb{Z})$, and define a stationary process $\{Q_n, n \in \mathbb{Z}\}$ by

$$Q_n = \max_{m \leq n} \left( -\sum_{j=m}^{n} \eta_j \right)^+.$$ 

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Note that $Q$ satisfies the Lindley recursion $Q_{n+1} = (Q_n - \eta_{n+1})^+$, and we have the following queueing interpretation. The number of customers in the queue at time $n$ is $Q_n$; if $\eta_{n+1} = -1$ a new customer arrives at queue and $Q_{n+1} = Q_n + 1$; if $\eta_{n+1} = 1$ and $Q_n > 0$, a customer departs from the queue and $Q_{n+1} = Q_n - 1$; otherwise $Q_{n+1} = Q_n$.

Note that the process $\eta$ can be recovered from $Q$, as follows:

$$\eta_n = \begin{cases} -1 & \text{if } Q_n > Q_{n-1} \\ 1 & \text{otherwise.} \end{cases} \quad (1)$$

For $n \in \mathbb{Z}$, set $\tilde{Q}_n = Q_{-n}$.

**Theorem 2** The processes $Q$ and $\tilde{Q}$ have the same law.

**Proof:** We first note that it suffices to consider a single excursion of the process $Q$ from zero. This follows from the fact that, at the beginning and end of a single excursion, the values of $\eta$ are determined, and so these act as regeneration points for the process. To see that the law of a single excursion is reversible, note that the probability of a particular excursion path depends only on the numbers of transitions (in the underlying Markov chain $\eta$) of each type which occur within that excursion path, and these numbers are invariant under time-reversal. \hfill \Box

Thus, if we define, for $n \in \mathbb{Z}$,

$$\hat{\eta}_n = \begin{cases} -1 & \text{if } Q_n > Q_{n+1} \\ 1 & \text{otherwise}, \end{cases} \quad (2)$$

we have the following corollary of Theorem 2.

**Corollary 3** The process $\hat{\eta}$ has the same law as $\eta$.

**Proof of Theorem 1:** Note that we can write $\hat{\eta}_n = \eta_{n+1} + 2(Q_{n+1} - Q_n)$. Summing this, we obtain, for $n \geq 1$,

$$\sum_{j=0}^{n-1} \hat{\eta}_j = \tilde{X}_n + 2(Q_n - Q_0). \quad (3)$$

where $\tilde{X}_n = \sum_{j=1}^{n} \eta_j$. If we adopt the convention that empty sums are zero, and set $\tilde{X}_0 = 0$, then this formula remains valid for $n = 0$. It follows that, on $\{Q_0 = 0\}$,

$$\sum_{j=0}^{n-1} \hat{\eta}_j = 2\tilde{M}_n - \tilde{X}_n, \quad (4)$$
where \( \bar{M}_n = \max_{0 \leq m \leq n} \bar{X}_m \).

Note also that, for \( m \in \mathbb{Z} \),

\[
Q_m = (Q_{m+1} - \hat{\eta}_m)^+ = \max_{n \geq m} \left( - \sum_{j=m}^{n} \hat{\eta}_j \right)^+.
\]  

(5)

The law of \( X \) conditioned to stay non-negative is the same as the law of \( \bar{X} \) conditioned to stay non-negative, since the events \( X_1 \geq 0 \) and \( \bar{X}_1 \geq 0 \) respectively require that \( \xi_1 = 1 \) and \( \eta_1 = 1 \), and so the difference in law between \( \xi \) and \( \eta \) becomes irrelevant. By Corollary 3, the law of \( \bar{X} \) conditioned to stay non-negative is the same as the law of the process

\[
\left( \sum_{j=0}^{n-1} \hat{\eta}_j, \ n \geq 0 \right)
\]

given that

\[
Q_0 = \max_{n \geq 0} \left( - \sum_{j=0}^{n-1} \hat{\eta}_j \right) = 0.
\]

By (4) this is the same as the law of \( 2\bar{M} - \bar{X} \) given that \( Q_0 = 0 \) or, equivalently, that \( \eta_0 = 1 \); but this is the same as the law of \( 2M - X \), so we are done. \( \square \)

In the queueing interpretation, \( \hat{\eta} = -1 \) whenever there is a departure from the queue and \( \hat{\eta} = 1 \) otherwise. Thus, Corollary 3 states that the process of departures from the queue has the same law as the process of arrivals to the queue; it can therefore be regarded as a non-Markovian analogue of the celebrated theorem in queueing theory, due to Burke [3], which states that the output of a stable \( M/M/1 \) queue in equilibrium has the same law as the input (both are Poisson processes). Note, however, that in this non-Markovian queueing process, the arrivals and services are not independent (being mutually exclusive). Our proof of Theorem 2 is inspired by the kind of reversibility arguments used often in queueing theory, although usually in a Markovian setting. For general discussions on the role of reversibility in queueing theory, see [2, 10, 17]; the idea of using reversibility to prove Burke’s theorem is originally due to Reich [16].

Finally, we remark that the following analogue of Theorem 1 holds in continuous time: let \( (\xi_t, \ t \geq 0) \) be a continuous-time Markov chain on \( \{-1, +1\} \) with \( \xi_0 = 1 \), and set \( X_t = \int_0^t \xi_s ds, \ M_t = \max_{0 \leq s \leq t} X_s \). We assume that the transition rates of the chain are such that event that \( X \) remains non-negative forever has positive probability. Then \( 2M - X \) has the same law as that of \( X \) conditioned to stay non-negative. The proof is identical to that of Theorem 1; in particular, the following analogues of Theorem 2 and Corollary 3 also hold: if we let \( (\eta_t, \ t \in \mathbb{R}) \) be a stationary version of \( \xi \) and, for \( t \in \mathbb{R} \), set

\[
Q_t = \max_{0 \leq s \leq t} \left( - \int_s^t \eta_s ds \right),
\]

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the $\bar{Q}$ (defined as $\bar{Q}_t = Q_{-t}$) has the same law as $Q$, and $\dot{\eta}$, defined by

$$
\dot{\eta}_t = \begin{cases} 
-1 & \text{if } \eta_t = 1 \text{ and } Q_t > 0 \\
1 & \text{otherwise,}
\end{cases}
$$

has the same law as $\eta$. The process $X$ in this setting is sometimes called the telegrapher's random process, because it is connected with the telegrapher equation. It was introduced by Kac [9], where it is also shown to be related to the Dirac equation. There is a considerable literature on this process and its connections with relativistic quantum mechanics (see, for example, [4, 5] and references therein).

For other variants and multidimensional extensions of Pitman’s theorem see [1, 7, 8, 11, 6, 12, 13, 14, 15] and references therein.

References


