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Introduction: We are motivated to study the quantification of entanglement for the basic reason that entanglement has been identified as a resource central to much of quantum information processing (see, e.g., Ref. [1]). To date, progress in the quantification of entanglement for mixed states has resided primarily in the domain of bipartite systems [2]. For multipartite systems in pure and mixed states, the characterization and quantification of entanglement presents even greater challenges.

As there are ambiguities in the generalizing of the entanglement of distillation (ED) and formation (EF) to multipartite systems [3], the study of other measures, such as the relative entropy of entanglement (ER), is thus relevant to understanding multipartite entanglement. Recently, a multipartite entanglement measure based on the geometry of Hilbert space has been proposed [4], and has been applied to several bipartite and multipartite cases. The merit of this measure is that it is suited for any-partite systems with any dimension, although determining it analytically for generic states remains a challenge.

One of the many test beds for entanglement measures includes a peculiar set of states: bound entangled states. These are states that are entangled, but from which no pure entangled state can be distilled if all parties are allowed only local operations and classical communication (LOCC). The entanglement of distillation (under LOCC) for these states is thus zero. Bound entangled states can be either bipartite or multipartite, the latter possibly exhibiting more features than the former. However, it does take nonzero amount of entanglement to create bound entangled states. It is thus desirable to see how much entanglement (e.g., ED) is needed to create them. So far, no analytic results on nontrivial entanglement measures for these states have been reported. Yet, as the two important measures—entanglement of distillation and of formation—have so far been limited to bipartite settings, in order to explore multipartite settings, it is, on the one hand, necessary to lay down bounds on the entanglement content for distillation and formation. On the other hand, applying other measures to multipartite states may prove helpful in understanding entanglement.

In this Letter, we study two distinct bound entangled states: Smolin’s four-party unlockable bound entangled state [5, 6] and Dür’s N-party Bell-inequality-violating bound entangled states [7]. For each, we determine analytically their geometric measure of entanglement, as well as a related quantity. These give lower bounds on their EF. In addition, we make conjectures on their relative entropies of entanglement. We have also found that when Smolin’s state is partitioned 1:234, it has bipartite $E_F = 1$, equal to that of a Bell state. Moreover, this bipartite mixed state can be reversibly transformed into a bipartite Bell state, and vice versa, even though they have different system entropies.

Geometric measure of entanglement: We begin by briefly reviewing the formulation of this measure. Consider a general, n-partite, pure state (expanded in the local bases $\{|e_i^{(i)}\}$): $|\psi\rangle = \sum_{p_1 \cdots p_n} \chi_{p_1 \cdots p_n} e_{p_1}^{(1)} e_{p_2}^{(2)} \cdots e_{p_n}^{(n)}$. As shown in Ref. [4], the closest separable pure state,

$$|\phi\rangle \equiv \otimes_{i=1}^n |\phi^{(i)}\rangle = \otimes_{i=1}^n \left( \sum_{p_i} c_{p_i}^{(i)} |e_{p_i}^{(i)}\rangle \right), \quad (1)$$

satisfies the stationarity condition

$$\sum_{p_1 \cdots p_n} \chi_{p_1 \cdots p_n}^{*} e_{p_1}^{(1)} \cdots e_{p_n}^{(n)} = \Lambda c_{p_1}^{(i)} \cdots c_{p_n}^{(i)}, \quad (2a)$$

(with $i = 1, \ldots, n$) and its complex conjugate, where the eigenvalue $\Lambda \in [-1, 1]$ is associated with the Lagrange multiplier enforcing $\langle \phi | \phi \rangle = 1$, and the symbol $^*$ denotes...
exclusion. Moreover, the eigenvalue $\Lambda$ is the cosine of the angle between $|\psi\rangle$ and $|\phi\rangle$; the largest one, $\Lambda_{\text{max}}$, which we call the entanglement eigenvalue, corresponds to the closest separable state, and is the maximal overlap:

$$\Lambda_{\text{max}}(|\psi\rangle) = \max_{\phi} \langle \phi | \psi \rangle,$$

where $|\phi\rangle$ is separable but otherwise arbitrary pure state. In Ref. [4], $E_{\text{sin}}^2 \equiv 1 - \Lambda_{\text{max}}^2(|\psi\rangle)$ was defined to be the geometric measure of entanglement for any pure state $|\psi\rangle$. In Ref. [8], $E_{\text{log}}(|\psi\rangle) \equiv -2 \log \Lambda_{\text{max}}(|\psi\rangle)$ was shown to be a lower bound on $E_R$ of $|\psi\rangle$.

The extension to mixed states can be built upon the pure-state theory, and is made via the use of the convex hull construction (indicated by “co”), as was done for $E_F$ [2]. The essence is a minimization over all decompositions $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$ into pure states:

$$E_{\text{sin}}^2(\rho) \equiv (\text{co} E_{\text{sin}}^2)(\rho) \equiv \min_{\{p_i, \psi_i\}} \sum_i p_i E_{\text{sin}}^2(|\psi_i\rangle),$$

$$E_{\text{log}}(\rho) \equiv (\text{co} E_{\text{log}})(\rho) \equiv \min_{\{p_i, \psi_i\}} \sum_i p_i E_{\text{log}}(|\psi_i\rangle).$$

This convex hull construction ensures that the measure gives zero for separable states; however, it also complicates the task of determining mixed-state entanglement. $E_{\text{sin}}^2$ is an entanglement monotone [4] and although it remains to see whether $E_{\text{log}}$ is also, it does, e.g., have the additive property:

$$E_{\text{log}}(|\psi_{12}\rangle \otimes |\phi_{34}\rangle) = E_{\text{log}}(|\psi_{12}\rangle) + E_{\text{log}}(|\phi_{34}\rangle).$$

Moreover, if the relationship among the two measures—the relative entropy of entanglement ($E_R$) and the entanglement of formation ($E_F$)—still holds for any multiparticle pure state: $E_R \leq E_F$, and if $E_F$ for mixed states is still constructed by the convex hull, then we have a lower bound on the entanglement of formation:

$$E_{\text{log}}(\rho) \leq E_F(\rho).$$

Using the inequality $-2 \log x \geq 1 - x^2$ (with $0 \leq x \leq 1$) for both base-2 and base-e, one further has

$$E_{\text{sin}}^2(\rho) \leq E_{\text{log}}(\rho).$$

We shall calculate analytically $E_{\text{sin}}$ and $E_{\text{log}}$ for the two bound entangled states, Smolin’s and Dür’s.

Smolin’s four-party unlockable bound entangled state: This is a four-qubit mixed state

$$\rho^{ABCD} = \frac{1}{4} \sum_{i=0}^{3} (|\Psi_i\rangle\langle \Psi_i|)_{AB} \otimes (|\Psi_i\rangle\langle \Psi_i|)_{CD},$$

where the $|\Psi_i\rangle$’s are the four Bell states: $((00) \pm (11))/\sqrt{2}$ and $((01) \pm (10))/\sqrt{2}$. Now, the state $\rho^{ABCD}$ can be conveniently rewritten as

$$\rho^{ABCD} = \frac{1}{4} \sum_{i=0}^{3} |X_i\rangle \langle X_i|,$$

where the $|X_i\rangle$’s are four orthogonal GHZ-like states:

$$|X_0\rangle = \frac{1}{\sqrt{2}}((0000) + (1111)), \ |X_1\rangle = \frac{1}{\sqrt{2}}((0011) + (1100)), \ |X_2\rangle = \frac{1}{\sqrt{2}}((0101) + (1010)), \ |X_3\rangle = \frac{1}{\sqrt{2}}((0110) + (1001)).$$

The most general decomposition of an arbitrary mixed state $\rho$ into pure states can be expressed as

$$\rho = \sum_{k=1}^{M} |\tilde{\varphi}_k\rangle \langle \tilde{\varphi}_k|,$$

where $M$ is an integer not smaller than $n$, the number of orthonormal eigenvectors $|\xi_i\rangle$ (without nonzero eigenvalues $\{\lambda_i\}$) of $\rho$, $|\tilde{\varphi}\rangle$’s are un-normalized, and $U$ satisfies $\sum_{k=1}^{M} U_{ki} U_{kj}^* = \delta_{ij}$. Thus, the most general (un-normalized) pure state that appears in the decomposition of Smolin’s state is

$$|\tilde{\varphi}_k\rangle = \frac{1}{\sqrt{2}} \sum_{i=0}^{n} U_{ki} |X_i\rangle.$$
We conjecture that the relative entropy of entanglement $E_R$ for this state is log 2 and that one of the closest separable mixed states is the state

$$\frac{1}{8}(|0000\rangle + |1111\rangle + |0011\rangle + |0101\rangle + |1010\rangle + |0110\rangle + |1100\rangle + |1000\rangle).$$

We remark that the negativity $N$ (defined as twice the absolute sum of negative eigenvalues of the partial transpose (PT) of the density matrix with respect to some bipartite partitioning) is zero for any $2/2$ partitioning, e.g., $\{AB : CD\}$, but nonzero for $1/3$ partitioning, e.g., $\{A:BCD\}$. Specifically, $N_{A:BCD} = 1$ but $N_{AB:CD} = 0$.

Dür’s $N$-party bound entangled state: Dür [7] found that for $N \geq 4$ the following state has bound entanglement:

$$\rho_N \equiv \frac{1}{N+1} \left( |\Psi_G\rangle \langle \Psi_G | + \frac{1}{2} \sum_{k=1}^{N} (P_k + \bar{P}_k) \right),$$

where $P_k \equiv |u_k\rangle \langle u_k|$ is a projector onto the state $|u_k\rangle \equiv \{0\}|0\rangle \cdots \{1\}|k\rangle \cdots \{0\}|N\rangle$, $P_k \equiv |v_k\rangle \langle v_k |$ projects onto $|v_k\rangle \equiv \{1\}|1\rangle \cdots \{0\}|k\rangle \cdots \{1\}|N\rangle$, and

$$|\Psi_G\rangle \equiv \left(|0\rangle^{\otimes N} + e^{in\pi/\sqrt{2}}|1\rangle^{\otimes N}\right)/\sqrt{2}. \quad (14)$$

For $N \geq 8$ this state violates the Mermin-Klyshko-Bell inequality for $N \geq 8$ [7]; violation was pushed down to $N \geq 7$ by Kaszlikowski et al. [9] for a three-setting Bell inequality; it was pushed further down to $N \geq 6$ by Sen et al. [10] for a functional Bell inequality. The phase $\alpha_N$ in $|\Psi_G\rangle$ can be eliminated by local unitary transformations, and hence we shall take $\alpha_N = 0$ in the following discussion.

In fact, if we consider the family of states

$$\rho_N(x) \equiv x|\Psi_G\rangle \langle \Psi_G | + \frac{1-x}{2N} \sum_{k=1}^{N} (P_k + \bar{P}_k), \quad (15)$$

we find that for $N \geq 4$ the state is bound entangled if $0 < x \leq 1/(N+1)$ and NPT entangled if $x > 1/(N+1)$. This can be seen from the fact that the negativities of the state $\rho_N(x)$ with respect to the two different partitions $(1:2 \cdots N)$ and $(2:3 \cdots N)$ are

$$N_{1:2 \cdots N}(\rho_N(x)) = \max \{0, [(N+1) \cdot x - 1]/N \}, \quad (16a)$$

$$N_{2:3 \cdots N}(\rho_N(x)) = x. \quad (16b)$$

By applying arguments similar to those used to calculate entanglement for Smolin’s state, we have that the general pure state in the decomposition of $\rho_N(x)$ has the form

$$\sqrt{y} e^{ik\theta} |\Psi_G\rangle + \sqrt{1-y} \sum_{k=1}^{N} \left( \sqrt{q_k} e^{ik\alpha} |u_k\rangle + \sqrt{r_k} e^{ik\beta} |v_k\rangle \right),$$

where $q$’s and $r$’s are non-negative and satisfy $\sum_k(q_k + r_k) = 1$. In this family, the state with the least entanglement (or maximum $\Lambda_{\max}$) for fixed $\{q_k, r_k\}$ is the one with all phase factors zero:

$$|\Psi(\{q, r\})\rangle \equiv \sqrt{y}|\Psi_G\rangle + \sqrt{1-y} \sum_{k=1}^{N} \left( \sqrt{q_k} |u_k\rangle + \sqrt{r_k} |v_k\rangle \right).$$

Next, we ask: For fixed $y$, what is the least entanglement that the above state can have? Take a separable state of the form $|\Phi\rangle \equiv \otimes_{i=1}^{N} (e^{i\alpha_i}|0\rangle + s_i|1\rangle)$; its overlap with $|\Psi(\{q, r\})\rangle$ is then

$$\langle \Psi|\Phi\rangle \equiv \sqrt{y}/2 (c_1 \cdots c_N + s_1 \cdots s_N) + \sqrt{1-y} \sum_{k=1}^{N} \left( \sqrt{q_k} c_1 \cdots s_k \cdots c_N + \sqrt{r_k} s_1 \cdots c_k \cdots s_N \right).$$

This can be shown to no greater than $\sqrt{(2-y)/2}$, again by a Schwarz inequality, taking

$$\{ \sqrt{y}/2, \sqrt{(1-y)/2}, \sqrt{(1-y)/2} \}$$

as the first $(2N+1)$-component vector (with modulus $\sqrt{(2-y)/2}$) and the corresponding coefficients as the second one (the modulus of which can be shown to be no greater than 1 for $N \geq 4$). The bound can be saturated, e.g., by

$$|\psi_{\pm, u,k}(y)\rangle \equiv \sqrt{y}|\Psi_G\rangle \pm \sqrt{1-y}|u_k\rangle, \quad (17a)$$

$$|\psi_{\pm, v,k}(y)\rangle \equiv \sqrt{y}|\Psi_G\rangle \pm \sqrt{1-y}|v_k\rangle, \quad (17b)$$

for which $\Lambda_{\max}(y) = \sqrt{(2-y)/2}$ [11]. As $1 - \Lambda_{\max}^2(y)$ is linear in $y$ and $-2 \log \Lambda_{\max}(y)$ is convex in $y$, one immediately gets

$$E_{\min}(\rho_N(x)) = \frac{x}{2}, \quad E_{\log}(\rho_N(x)) = \log \frac{2}{2-x}, \quad (18)$$

and one of the optimal decompositions is

$$\rho_N(x) = \frac{1}{4N} \sum_{k=1}^{N} \sum_{\alpha, \beta = u, v} |\psi_{\alpha, \beta,k}(x)\rangle \langle \psi_{\alpha, \beta,k}(x) |. \quad (19)$$

We conjecture that, for $N \geq 4$, $\rho_N(x)$ has $E_R(x) = x \log 2$, with one closest separable mixed state being

$$\frac{x}{2} \left( |00\rangle \langle 00 | + |11\rangle \langle 11 | \right) + \frac{1-x}{2N} \sum_{k=1}^{N} (P_k + \bar{P}_k).$$

The Mermin-Klyshko-Bell inequality, $\text{Tr}(B_N p_N) \leq 1$ for local hidden variable theory, can be violated by $\rho_N(x)$ for certain values of $x$ and $N$ as $\text{Tr}(B_N p_N(x)) = 2^{(N-1)/2} x$. Similar results can be obtained for two other inequalities [9, 10].

Concluding remarks: We have analytically determined the geometric measure of entanglement and a related
logarithmic quantity for two bound entangled states. These two quantities provide lower bounds on entanglement of formation. We have also made conjecture on their relative entropies of entanglement (which are at least upper bounds). Assuming $E_D \leq E_H$ holds, we then also have upper bounds on distillable entanglement for these states.

For Smolin’s bound entangled state, we can establish its entanglement of formation for certain bipartite partitions. For example, if we group the four parties ABCD in two, A:BCD, we can write the state as

$$
\rho^{A:BCD} = \frac{1}{4} \sum_{i=0}^{3} |X_i\rangle\langle X_i|,
$$

with the 3-qubit states of BCD mapped on to the 8-level system (000 → 0, 001 → 1, ..., 111 → 7) and arrive at the convertible states (by BCD)

$$
|X_0\rangle = (|00\rangle + |11\rangle)/\sqrt{2}, \quad |X_1\rangle = (|01\rangle + |10\rangle)/\sqrt{2},
$$

$$
|X_2\rangle = (|02\rangle + |12\rangle)/\sqrt{2}, \quad |X_3\rangle = (|03\rangle + |13\rangle)/\sqrt{2}.
$$

In order to find the entanglement for this bipartite state (of $C^2 \otimes C^8$), we need to consider the entanglement of the general pure state (properly normalized)

$$
|\psi\rangle = \sum_{i} \sqrt{\bar{\nu}_i} e^{i\phi_i} |X_i\rangle
$$

that appears in the pure-state decompositions. In fact, regardless of the values of the $x_i$’s, this pure state has a reduced density matrix (tracing over BCD) of the form $(|0\rangle\langle 0| + |1\rangle\langle 1|)/2$. This shows that the mixed state $\rho^{A:BCD}$ has $E_P = 1$, $E_{\text{ein}} = 1/2$, and $E_{\text{log}} = 1$. In fact, this is a general result for $C^2 \otimes C^{2m}$ states that are derived from mixing Bell-like states

$$
|\Psi_2^k\rangle \equiv (|0, k\rangle \pm |1, 2m-k-1\rangle)/\sqrt{2},
$$

having distinct $k$’s, where $k = 0, 1, \ldots, m-1$.

The above considerations indicate that the bipartite state $\rho^{A:BCD}$ can be created by starting with an ebit shared between A and BCD. Suppose that they start in a bipartite state $|X_0\rangle$. The party BCD sends the state $|X_0\rangle$ into a local apparatus that leaves the state unchanged (with probability 1/4), or locally transforms $|X_0\rangle$ to $|X_1\rangle$ (via $0 \rightarrow 3, 7 \rightarrow 4$, etc.), $|X_2\rangle$, and $|X_3\rangle$ (with probabilities 1/4 each) respectively. This protocol achieves the mixed state $\rho^{A:BCD}$ from a bipartite ebit. On the other hand, the two parties can distill a bipartite Bell state using the protocol proposed by Smolin [5, 6], in which CD (as BCD are now in the same party) make a Bell measurement and pass the result to A and B such that they establish a Bell state. Thus we have found a bipartite mixed state that can be reversibly converted into a Bell state, even though their system entropies are different.

Although Dür’s bound entangled state violates a Bell inequality, it has nonzero negativity under certain partitions. One may raise the question: Does there exist a bound entangled state that is PPT under all partitions but that still violates a Bell’s inequality? For example, does a UPB bound entangled state [12] violate a Bell inequality? We shall see shortly that the answer is “No”, at least for the three different Bell inequalities [7, 9, 10] mentioned earlier. Acín has shown [13] that if an $N$-qubit state violates a two-setting Bell inequality then it is distillable under certain bipartite partitioning. Using the results of Ref. [14] regarding distillability, one can repeat the same analysis for the other two Bell inequalities [9, 10] and obtain the same conclusion. This bipartite distillability then implies a negative partial transpose (NPT) under that bipartite partitioning according to Horodecki et al. [15] Hence, violating these Bell inequalities implies NPT under certain bipartite partitioning. Said equivalently, if a state has positive partial transpose (PPT) under all bipartite partitions then the state never violates these Bell inequalities.

So far, the smallest $N$ for which Dür’s bound entangled state violates a Bell inequality is $N = 6$. Can the number be further pushed down to 4 by some other forms of inequalities? If so, can Dür’s $N = 4$ and $N = 5$ bound entangled states be distillable under some bipartite partitioning? These questions remain open.

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Appendix A: In this Appendix we sketch proofs of two useful inequalities and describe the derivation of the entanglement eigenvalue for the states in Eqs. (17). We start with the first sought inequality:

$$
(c_1 c_2 c_3 c_4 + s_1 s_2 s_3 s_4)^2 + (c_1 c_2 s_3 s_4 + s_1 s_2 c_3 c_4)^2 + (c_1 s_2 s_3 s_4 + s_1 c_2 s_3 s_4)^2 + (c_1 s_2 c_3 s_4 + s_1 s_2 c_3 s_4)^2 \leq 1.
$$

We have simplified the notation by using $c_i \equiv \cos \theta_i$, and $s_i \equiv \sin \theta_i$. By subtracting the left-hand side from 1 and making some algebraic manipulation, we arrive at the non-negative expression (hence the sought result):

$$
(c_1 c_2 c_3 s_4 - s_1 s_2 s_3 s_4)^2 + (c_1 c_2 s_3 s_4 - s_1 s_2 c_3 s_4)^2 + (c_1 s_2 c_3 c_4 - s_1 c_2 s_3 s_4)^2 + (c_1 s_2 c_3 s_4 - s_1 s_2 c_3 c_4)^2.
$$

The next sought inequality is (for $N \geq 4$):

$$
f_N \equiv (c_1 \cdots c_N + s_1 \cdots s_N)^2 + \sum_{k=1}^{N} ((c_1 \cdots s_k \cdots c_N)^2 + (s_1 \cdots c_k \cdots s_N)^2) \leq 1.
$$

First, making similar arguments, one can show that $f_4 \leq 1$. One can also show that $f_N+1 \leq f_N$. Thus by induction, we have the sought result.

We now discuss why $\sqrt{g}|\Psi_G\rangle = \pm \sqrt{1 - y}|\psi_k\rangle$ and $\sqrt{g}|\Psi_G\rangle = \pm \sqrt{1 - y}|\psi_k\rangle$ have as their maximal entanglement...
eigenvalue $\lambda_{\text{max}}(y) = \sqrt{(2-y)/2}$. As one can make local relative phase shifts to transform $\sqrt{y}/|\Psi_G \rangle + \sqrt{1-y}/|u_k \rangle$ to $\sqrt{y}/|\Psi_G \rangle - \sqrt{1-y}/|u_k \rangle$, they have the same entanglement. The change from $\sqrt{y}/|\Psi_G \rangle \pm \sqrt{1-y}/|u_k \rangle$ to $\sqrt{y}/|\Psi_G \rangle \pm \sqrt{1-y}/|u_k \rangle$ is simply a flipping of 0 to 1, and vice versa. The mapping from $k$ to $k'$ is just a relabelling of parties. Thus, we need only consider the state
\[
\sqrt{y/2} (|00\cdots0\rangle + |11\cdots1\rangle) + \sqrt{1-y}|10\cdots0\rangle.
\]
As this state is invariant under permutation of all parties except the first one, and as the coefficients are non-negative, in order to find the maximal overlap we can make the hypothesis that the closest separable state is of the form
\[
\left( \sqrt{p}(0) + \sqrt{1-p}(1) \right) \otimes \left( \sqrt{q}(0) + \sqrt{1-q}(1) \right)^{\otimes N-1}.
\]
We further see that in order for the overlap to be maximal, $q$ must be either 1 or 0. For the former case, we can further maximize the overlap to get $\sqrt{(2-y)/2}$. For the latter case, the maximum overlap is $\sqrt{y/2}$ (as $0 \leq y \leq 1$). Hence, the state $\sqrt{y}/|\Psi_G \rangle \pm \sqrt{1-y}/|u_k \rangle$ has the entanglement eigenvalue $\sqrt{(2-y)/2}$.

Appendix B: In this Appendix we analyze the connection between violation of three Bell inequalities and bipartite distillability as was done in Ref. [13] for the two-setting inequality. It was shown by Dür and Cirac [16] that an arbitrary $N$-qubit state $\rho$ can be locally depolarized into the form
\[
\rho_N = \lambda_0^+ |\Psi_0^+ \rangle \langle \Psi_0^+ | + \lambda_0^- |\Psi_0^- \rangle \langle \Psi_0^- | + \sum_{j=1}^{2^{N-1}-1} \lambda_j (|\Psi_j^+ \rangle \langle \Psi_j^+ | + |\Psi_j^- \rangle \langle \Psi_j^- |),
\]
while preserving $\lambda_0^+ = \langle \Psi_0^+ | \rho |\Psi_0^+ \rangle$ and $\lambda_j = \langle \Psi_j^+ | \rho |\Psi_j^+ \rangle + \langle \Psi_j^- | \rho |\Psi_j^- \rangle$, where $|\Psi_0^+ \rangle \equiv (|0^\otimes N \rangle \pm |1^\otimes N \rangle)/\sqrt{2}$, and the $|\Psi_j^\pm \rangle$'s are GHZ-like states, i.e., the states in (22), unfolded into qubit notation. Normalization gives the condition
\[
\lambda_0^+ + \lambda_0^- + 2 \sum_j \lambda_j = 1.
\]
Now define $\Delta \equiv \lambda_0^+ - \lambda_0^-$, which we assume to be non-negative (w.o.g). The condition that there is no bipartite distillability for some bipartite partitioning $P_j$ is [14]
\[
2\lambda_j \geq \Delta.
\]
Assuming non-distillability for all bipartite splittings, we have
\[
2 \sum_j \lambda_j = 1 - (\lambda_0^+ + \lambda_0^-) \geq (2^{N-1} - 1)\Delta.
\]
As $\lambda_0^+ + \lambda_0^- \geq \Delta$, we have further that
\[
1 - \Delta \geq (2^{N-1} - 1)\Delta. \tag{23}
\]
For the two-setting Bell inequality considered by Acín [13], violation implies
\[
\Delta > 1/2^{(N-1)/2}.
\]
For the three-setting Bell inequality considered in [9], violation implies
\[
\Delta > \sqrt{3}(2^{N}/3^{N}).
\]
For the functional Bell inequality in [10], violation implies
\[
\Delta > 2(2^{N}/\pi^{N}).
\]
One can easily check that the three Bell inequalities considered are inconsistent with non-bipartite-distillability condition, Eq. (23). Hence, the violating of these three Bell inequalities implies the existence of some bipartite distillability.

[2] For a review, see M. Horodecki, Quant. Info. Comp. 1, 3 (2001), and references therein.
[11] In arriving at this result we have made a hypothesis about the form of the closest separable pure state. Whilst we believe the result to be true, we do not have a rigorous proof. But we have verified numerically for the $N = 4$ case.