Reductionism isn't Functional

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One of the great current debates in biology concerns whether the observed behaviour of a system can be accounted for in terms of the behaviours of its subcomponents. The problem is often presented as a question of 'holism versus reductionism'; whether 'the whole is greater than the sum of its parts'. The holist position is that biological systems display certain phenomena that cannot be understood by thinking about the system at any lower level than its entirety; reductionists argue that systems can be explained completely in terms of their subcomponents. We show that de Simone's theorem, a result derived in a relation of Professor Robin Milner (FRS)'s Turing Award winning work in concurrency theory (a branch of theoretical computer science) provides an answer to this problem. De Simone's theorem proves that all possible systems can be reasoned about in terms of their subcomponents. Hence, if the parts of systems are represented as processes with internal state rather than functions, it is always possible to explain system behaviour in terms of the interactions between its parts.

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Summary

One of the great current debates in biology concerns whether the observed behaviour of a system can be accounted for in terms of the behaviours of its subcomponents. The problem is often presented as a question of 'holism versus reductionism'; whether 'the whole is greater than the sum of its parts'. The holist position is that biological systems display certain phenomena that cannot be understood by thinking about the system at any lower level than its entirety; reductionists argue that systems can be explained completely in terms of their subcomponents. We show that de Simone's theorem, a result derived in a relation of Professor Robin Milner(FRS)'s Turing Award winning work in concurrency theory (a branch of theoretical computer science) provides an answer to this problem. De Simone's theorem proves that all possible systems can be reasoned about in terms of their subcomponents. Hence, if the parts of systems are represented as processes with internal state rather than functions, it is always possible to explain system behaviour in terms of the interactions between its parts.

1 Introduction

Concerns over reductionism have been raised in many areas of biology (Williams, 1997), including ecology (Bergandi & Blandin, 1998; Lenton, 1998; Levin, 1998; Wilkinson, 1999), evolution (Stebbins & Ayala, 1981; Gould, 1998; Wilson, 1997; Seaborg, 1999), neuroscience (Edelman & Tononi, 1995; Barlow, 1998), behaviour (Gould, 1997; Rose, 1998; Goodwin, 1998a) and developmental biology (Berrill & Godwin, 1996; Alberch & Branco, 1998; Brenner, 1998). Frequently-cited reasons for the failure of reductionism include (1) it cannot cope with contingent behaviour or code up a system's *history*; (2) it gives insufficient weight to *interactions* between subcomponents; and (3) it cannot deal with *emergent properties* that seem not to be explicable in terms of subcomponents alone (Williams, 1997; Mayr 1985). However, these claims result from associating the activity of reducing a system to its sub-components with the mathematical representation of those entities. Hence most biologists might claim to accept the notion of constitutive reductionism[1], but many do not accept that the properties or behaviours of constitutively reducible systems can be deduced from a knowledge of the properties of their components (Mayr, 1998; Williams 1997). For instance, Ernst

[1]Biologists and philosophers (e.g., Mayr, 1982, Sarkar 1992) distinguish between *theory* (where a theory or branch of science is shown to be consistent with, derived from or explained by another); *explanatory* (where a system is explained in terms of the behaviour of more fundamental systems or components) and *constitutive* reductionism (the principle that systems are composed of systems or entities at a lower level and conform to the laws governing the latter). The interesting debates concern theory or explanatory reductionism; constitutive reductionism (i.e. that biological systems are composed of molecules and atoms and these systems obey the laws of physics/chemistry) is generally accepted.

Mayr (1985) argues (p 58): "[biological] Systems at each hierarchical level have two characteristics. They act as wholes (as if they were a homogenous entity), and their characteristics cannot (not even in theory) be deduced from the most complete knowledge of the components, taken separately or in other partial combination".

Many of the limitations stated for the reductionist approach result not from thinking about the system in terms of subcomponents, but of representing those subcomponents as *functions*. For instance, May (1998) states "Much of the reductionist success in physics can be put down to the existence of linear superposition principles; one can dissemble things and then meaningfully reconstruct the whole by adding them back together. When you have non-linear phenomena, processes similar to phase transitions in physics occur, where small changes in one variable can make for discontinuous changes in the whole system. Phenomena such as these are difficult to intuit from the study at the lower level." This has lead some biologists to take extreme positions: "Most of what is most characteristic of living organisms cannot be expressed in mathematical terms or in terms of the simplistic laws of physics" (Mayr, 1985; p 54). The success of functional representation of physical and biological systems has been tremendous and students of these areas may be tempted to regard this as the only 'proper' mathematical representation of a system. One consequence of this view is the use of increasingly complex functions to account for a biological system's behaviour (Goodwin *el al.*, 1989). As the complexity increases, the mapping between the functions used and the recognised components within the system is lost. Interactions between components are also difficult to represent explicitly: rather, they are implicit in the coupling between equations. Since the system is no longer represented (and hence understood) in terms of its components and interactions it is believed that this implies that it cannot be reduced. In fact, this is the consequence of using functions to account for the behaviour or effects of the sub-components, rather than an approach that allows natural representation (i.e., every object in the system is represented by an object in the model) of those sub-components. When biological complexity is considered from a computer science theory perspective, and interactions are modelled explicitly, hierarchy and emergence appear less problematic for reductionism (Muir, 1982); a position taken by Brenner (1998, p110): "We do not have to talk about these properties as emergent but look on them as arising directly from the properties of their components and their interactions. The whole is some special mathematical function of the parts, but we can safely say that this function is not the sum".

2 A Modelling Approach

The need to reason about concurrent computer systems, which are intrinsically composed of many interacting parts, has provided a major challenge to the ways we represent and analyse the dynamics of complex systems. This need has given rise to an algebraic approach to reasoning over systems formed from component parts that treat those parts as first class objects. In the process algebraic approach (Milner 1980, 1983, 1989; Hoare, 1985; Hennessy, 1988; Baeten & Wiejland, 1990; Baeten, 1990) systems are represented as compositions of *processes* that have internal state, and interact by exchanging messages. The addition of state means that, unlike functions, when given the same arguments (input message(s)) a process need not always give the same response (results/outputs). A particularly important consequence of the process view of systems is that all[2] possible types or patterns of interaction can be represented (de Simone, 1985) with a limited and therefore comprehensible set of operators.

In reasoning about a system the important issue is to compose (or add) the parts together correctly. In other words, we need appropriate operators to 'glue' our components together. In a system formed from interacting parts this will inevitably be some form of parallel composition (see Box 1), since we must permit the parts to coexist simultaneously. With the operators *action prefix*, *parallel composition*, *restriction*, *recursion* and *choice* of the process algebra SCCS/Meije, de Simone (1985) proved that any feasible parallel operator can be represented, and hence we can represent any way of composing a system. That is, no matter what the nature of the interactions which dictate the behaviour of the composed system as a result of the behaviour of its parts, they can be represented by the five basic operators of SCCS/Meije. De Simone's result implies that in order to understand any composed system we need only understand the representation of its underlying compositions within SCCS/ Meije.

Explaining the operators in slightly more detail, see Milner (1989) for a tutorial on process algbera, a process term is presented with basic operators which act as follows:

> Action prefix: we write $a : P$ to describe a process that performs the action a and as a consequence becomes the process P. This action can be any suggestive name. For instance we might use *breed* and write $breed : MoreAnimals$. As a convention we tend to start action names with a small letter and process names with a capital to distinguish between them. Using the later operators we can see how

[2]Strictly all systems formed from bisimulation (Milner, 1980, 1989) preserving operators. Bisimulation preservation is a very weak requirement which phisical compositions will obey. However, in a formal logical context it is possible to derive abstract operators that will not preserve bisimulation. Stronger demonstrations of the generality of process algebra, which do not require bisimulation preservation are achieved by defining the scope (Groote & Vaandrager 1992; Bloom *et al.*, 1995) of the operational rules permitted in the definition of a particular calculus.

we might sensibly form the process $MoreAnimals$.

Choice: we write $P + Q$ to define a process that can be *either* the process P or the process Q. As an example $eat : Live + starve : Die$ will denote a process that chooses to perform the actions *eat* or *starve* and evolves respectively into the processes *Live* or *Die*. Notice at this point we have given no information about how the choice is resolved, clearly we shall wish that this choice can be influenced by the environment, which in this instance will be other processes.

Parallel composition: we write $P \times Q$ to represent the process P executing alongside the process Q in synchronous parallel. As an example $eat : Live \times eat : LIve$ might define two animals living in the same environment needing to eat to live. The importance of this being a *synchronous* parallel is that we insist that both of the processes evolve at the same time, in order that the compound may itself evolve.

Permission: we write $P\lceil S$ to say that only those actions in a set names S performed by P are permitted to occur. For example $(eat : Live + stuff : Happy)\lceil\{eat\}$ is not allowed to perform the action $stuff$. So far we have no indicated how one process influences another, only that they should be capable of such. If we choose our to structure our actions wisely so that they come in pairs *input* and *output* which we can denote by a and $\overline{a}$ and further arrange that when these two actions are performed at the same time then they in some sense 'vanish'[3]. The combination of actions structured in this way and permission allows us to *insist* that certain actions 'vanish' and this can only occur if they communicate (the combination of input and output) with their partner within the scope of such a permission. It is this combination of effects that gives the process algebra its descriptive power.

Recursion: since we will want our processes to have the capability to persist we need a notion of loop. The simples way to introduce this is to allow process variables and then bind process expressions to them, with the intent that whenever we see a variable we replace it with its associated definition. Writing $P \stackrel{def}{=} E$ for the act of defining the variable name[4]. As an example a simple creature $Animal \stackrel{def}{=} eats : Animal + starves : 0$, this is an animal which if it *eats* continues otherwise it *starves* and is then capable of no further action which is given as a process written 0.

[3]For the technically minded the most elegenat way to do this is to draw our names from a free abelian group and insist that inverses represent the dual output to the input. The identity element in the group is now the natural representation of the passage of one unit of time or computation with no observable behaviour taking place.

[4]In this case we have to be careful that definition and equality are not confused. In the full presentation of a process algebra we have a formal notion of what it is for two processes to be equal and we reserve the equals sign for this.

In the full presentation (Milner, 1983; deSimone, 1985; Milner, 1989) the meanings of the operators above, which are expressed as intent in english herein, are fornally defined via a natural deduction style recursively over the syntax which defines the process.

Whilst process calculi such as SCCS deal only with causality and time and are therefore limited in their ability to account for physical phenomena, they can be simply extended to include phenomena such as probability or priority (Tofts, 1994), and consequently have sufficient expressive power to represent and reason (Tofts, 1992) over many complex systems, including biological ones. They have been used successfully to model aspects of social insect behaviour (Tofts 1992), host-parasite interaction (Christodolou, 1999) evolution of sex determination (Hatcher & Tofts, 1995), music composition (Ross, 1995) and interacting computer systems (Milner, 1989).

One of the most fully realised component views of systems is the recently popular object oriented programming approach. This approach to system description has been exploited widely in the modelling of individual based systems within the biological community. Indeed the original notation *Simula* (Birtwistle *et al.*, 1979) was intended by its author to be a physical system description language, "which with the addition of input/output statements *could* be executed on a computing system". Process algebra provides a natural (fully abstract with one-to-one mapping of components) model of such languages. Recent work in this area (Milner *et al.*, 1992) seems to indicate a world view which may well be more fundamental than that of the lambda calculus (Barendegt, 1987), the formal presentation of functions.

The SCCS/Meije observation that all possible parallel operators are representable implies that, if one accepts constitutive reduction, *any* system can *always* be represented in terms of its parts. Therefore, in principle, all systems can be explained in terms of the sum (composition) of their parts. Whether a process-based approach provides adequate explanation for the behaviour of a system will depend, as with any other modelling paradigm, on the development of models and their scientific test. By allowing components to have different states, models in which the behaviour of parts and hence systems are contingent on history are automatically obtained. By shifting the emphasis to objects (components) rather than functions, interactions are also naturally representable. Hence, with a state-based approach, objections (1) and (2) are not true of reductionism, if by reduction we mean the act of abstrating or describing a system in terms of it parts.

3 Discussion

The suggestion that emergence illustrates the failure of reductionism is flawed from a mathematical perspective. Emergent phenomena are a frequent observation from all disciplines of biological modelling: once a model is built, complex (sometimes unexpected and often counter-intuitive) patterns of behaviour may emerge. This is true even within a functional framework: cyclic or chaotic dynamics may 'emerge' from functional representations of populations, yet clearly this complex dynamical behaviour is not, in any meaningful sense, more than the 'sum' of its parts. Similarly, the object-oriented community has frequently observed that complex patterns emerge from relatively few object types interacting with few and simple rules (developmental biology: Kersberg & Changeux, 1998; Kersberg & Wolpert, 1998; social insect behaviour: Tofts, 1992; Bonabeau *et al.*, 1998). The models involved are reductions of observed systems to subcomponents and interactions, yet these reductionist tools show emergent properties. Hence, the mere observation that system-level behaviour appears complicated and is *currently* inexplicable within a reductionist paradigm does not imply that only alternative paradigms can succeed (Sarkar, 1992).

Reductionism is at a disadvantage because the term is seldom defined explicitly, and most frequently it is discussed with reference to its perceived failings by those that believe it cannot work. For instance, Capra (1996) (p 17) defines reductionism implicitly by comparison of terms: "The emphasis on the parts has been called mechanistic, reductionist, or atomistic; the emphasis on the whole holistic, organismic, or ecological". With strict interpretation of criteria (1) -(3), most non-trivial biological systems would appear to be holistic; indeed Capra cites non-linear dynamics, chaos and fractals, network dynamics, self-organisation and quantum physics as examples of holist theory. However, under strict holism, wholes are regarded as inexplicable in terms of parts, thus every system is considered indivisible or atomistic. Hence the notions of 'parts' and 'levels' of greater or lesser inclusion are rendered meaningless, and the concept of 'atom' changes for each system considered. Such a methodology denies the attempt to explain any system in terms of others, and restricts understanding to observation alone; it therefore must lie outwith the practice of science.

Apparent holism is particularly common in the popular science literature and popular press (Lovelock, 1991; Eldredge, 1995; Polkinghorne, 1996), and is sometimes used (or abused) to imply that reductionist science is harmful or dangerous (Rose, 1997; Volk 1997; Turney, 1998; Rifkin, 1998). For instance, Capra (1996, p5): argues that species extinctions, population explosion, third world debt and ethnic violence are "just different facets of one single crisis, which is a crisis of perception" (p 5). Capra (p6) suggests that solution to these problems requires a "paradigm shift" from "outmoded" reductionist thinking to "a holistic worldview, seeing the world as an integrated whole rather than a dissociated collection of parts". Goodwin (1998b) reviews Turney (1998) agreeing that reductionist science has led to disturbing practices including vivisection, eugenics, genetic engineering, *in vitro* fertilisation and 'designer babies', and also appends his own list of "other biological nightmares" (p 49); suggesting "continuing public anxiety about the practice and applications of science should be read as a correct diagnosis of a pathology in the type of science we pursue" (p 49). These and other concerns are also raised in Appleyard (1999), Rifkin (1998), Scheider (1996) and are countered in Wolpert (1999).

Whilst the question of holism over reductionism might seem esoteric or outmoded, it is of fundamental importance because its debate in the public domain reflects and influences the public perception of science. The limitations of reductionism are not those imposed by thinking of systems in terms of their parts, but by the way we represent and analyse those parts. A system need never be greater than the sum of its parts, if one does the sums correctly.

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Box 1

The formal functional representation of the natural numbers due to Church (Barendregt, 1987) is well known. By way of contrast there are many ways to construct the natural numbers as processes or objects, for simplicity we use CCS (Milner 1980, 1989). The example below shows that we represent both the objects (in this case numbers) and the glue (operators) between them as processes. By convention names starting with a lower case letters are *actions* (with overbarring denoting the output duals of the default input behaviour), and Names starting with an upper case letter are *states.* The operators: | is parallel composition, + is choice, . is action prefix, $[a/b]$ renames action a by action b and $\backslash\{a,\ldots\}$ denotes restriction or context closure, see (Milner 1989) for the formal presentation of this reasoning methodology. For those unused to formal representations of the natural numbers the below may seem cumbersome, but in fact is no harder, indeed in many ways simpler than the standard functional approaches (Barendregt, 1987).

$$
\begin{array}{rcl}
DZero & \stackrel{def}{=} & \overline{done}.DNil \\
DOne & \stackrel{def}{=} & \overline{one}.DZero \\
DTwo & \stackrel{def}{=} & one.DOne \\
DThree & \stackrel{def}{=} & one.DTwo \\
\vdots & \vdots & \vdots \\
DN & \stackrel{def}{=} & \overline{one}.(DN-1) \\
Three & \stackrel{def}{=} & (copy.Three|DThree) \\
N & \stackrel{def}{=} & (copy.DN|N) \\
Do\_Add & \stackrel{def}{=} & copy.\overline{copy1}.\overline{copy2}.done1.done2.\overline{done}.Do\_Add \\
Add(N,M) & \stackrel{def}{=} & (N[copy1/copy, done1/done]|M[copy2/copy, done2/done]| \\
 & & Do\_Add)/\{copy1, copy2, done1, done2\} \\
Do\_Minus & \stackrel{def}{=} & copy.\overline{copy1}.\overline{copy2}.Doing\_M \\
Doing\_M & \stackrel{def}{=} & onel.(oner.Doing\_M + donel.Done\_M1) + donel.Done\_M1 + doner.Done\_M \\
Done\_M1 & \stackrel{def}{=} & onel.one.Done\_M1 + doner.Done\_M \\
Done\_M & \stackrel{def}{=} & \overline{done}.Do\_Minus \\
Sub(N,M) & \stackrel{def}{=} & (N[copy1/copy, donel/done, onel/one]|M[copy2/copy, done2/done, oner/one]| \\
 & & Do\_Add)/\{copy1, copy2, done1, done2\}
\end{array}
$$

In the above we define addition and subtraction; it is similarly straightforward to define multiplication and division in this style. It should be noted that by inheriting state from the numbers involved in the operators, the operators in this view do not have to count themselves, and therefore we have not used the power of the natural numbers to represent operators over them, as one would hope. Indeed one can demonstrate that functions are just special types of process that do not have internal state. Hence in some circumstances it is possible to abstract from a process theoretic system description to a functional one.