Products of Sines in Two Simple Arrangements of Six Lines

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Abstract

We show that in two different arrangements of six lines in the Euclidean plane an inequality holds between the products of the sines of selected angles from the arrangement. Either of these then provides a short proof of the falsity of Ringel's conjecture, using no more than schoolbook geometry, as opposed to the oriented matroid techniques of Las Vergnas.

1. Introduction

Ringel [5] conjectured that in an arrangement of lines in general position the slopes could be arbitrarily prescribed. This conjecture was disproved first by Las Vergnas [3] using oriented matroid techniques over a 32 point dual construction. Richter and Sturmfels [4] improved this to give a 6 line counterexample (figure 1), still demonstrating the slope constraint using oriented matroid techniques. Felsner and Zieglar [2] give a different proof of the counterexample using higher Bruhat orders. In contrast, we directly analyse the figure using schoolbook geometry.

2. Products of Sines

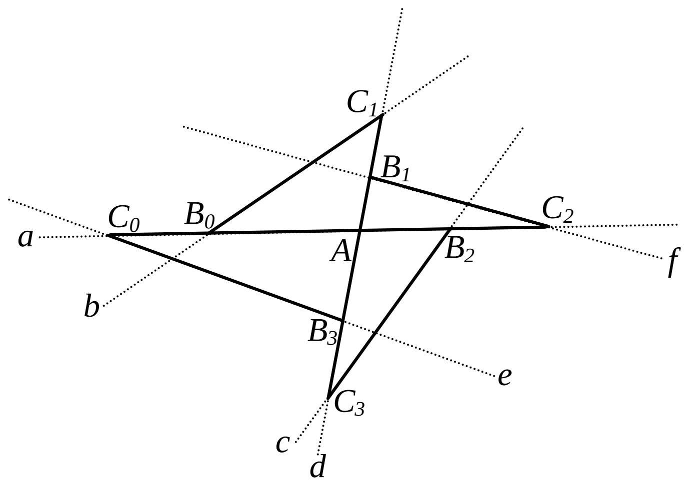

Figure 1

Theorem: in figure 1, with C_4=C_0, $\prod_{i=0}^{3} \sin(\angle AB_iC_{i+1}) > \prod_{i=0}^{3} \sin(\angle AC_{i+1}B_i)$.

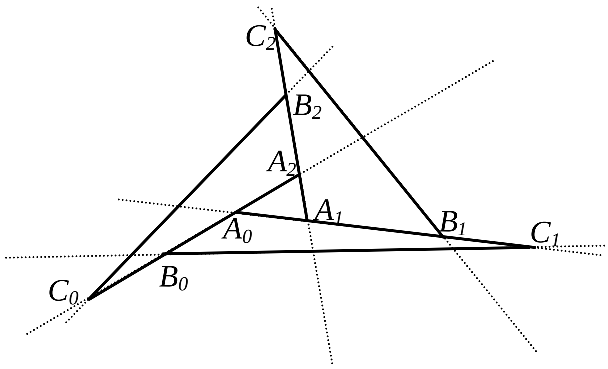

Figure 2

Theorem: in figure 2, with $C_3=C_0$, $\prod_{i=0}^{2} \sin(\angle A_i B_i C_{i+1}) > \prod_{i=0}^{2} \sin(\angle A_i C_{i+1} B_i)$.

Proof: In the first figure, take $A=A_0=A_1=A_2=A_3$, and $B_4=B_0$, take subscripts i ranging from 0 to 3. In the second figure, take subscripts i ranging from 0 to 2, and take $A_3=A_0$ and $B_3=B_0$. We have: $0 < |A_{i+1}B_{i+1}| < |A_i C_{i+1}|$. Taking products:

$$\prod |A_i B_i| < \prod |A_i C_{i+1}|$$

By the sine formula, for the highlighted triangles: $\dfrac{\sin(\angle A_i B_i C_{i+1})}{|A_i C_{i+1}|} = \dfrac{\sin(\angle A_i C_{i+1} B_i)}{|A_i B_i|}$.

A substitution gives the results.

In [1], these results are generalized to n triangles exscribed around a convex polygon with n sides.

3. Disproving Ringel's Conjecture

In the first figure, the angles to the horizontal of the lines *a, b, c, d, e* and *f* are approximately: 0°, 35°, 55°, 80°, 160° and 165°, respectively. If we could draw the figure with the lines at angles 0°, 15°, 20°, 100°, 125° and 145° respectively, then we would contradict the first theorem. A similar argument holds for the second figure.

4. References

[1] J.J. Carroll, The Sharpness of Circular Saws, HP Labs Technical Report, HPL-2000-74, 2000.

[2] S. Felsner and G.M. Zieglar, Zonotopes associated with higher Bruhat orders, Discrete Mathematics 241 (1-3), 2001, (301-312)

[3] M. Las Vergnas, Order properties of lines in the plane and a conjecture of G. Ringel, J. Combinatorial Theory, Ser. B., 41, 1986, (246-249).

[4] J. Richter and B. Sturmfels, On the topology and geometric construction of oriented matroids and convex polytopes, Transactions Amer. Math. Soc. 325, 1991, (389-412).

[5] G. Ringel, Teilungen der Ebene durch Geraden oder topologische Geraden, Math. Zeitschrift, 64, 1956, (79-102).