Stronger Topologies for Sample Path Large Deviations in Euclidean Space

Neil O'Connell
Basic Research Institute in the Mathematical Sciences
HP Laboratories Bristol
HPL-BRIMS-96-05
February, 1996

In this paper we present sufficient conditions for sample path large deviation principles to be extended to finer topologies. We consider extensions of the uniform topology by Orlicz functionals and we consider Lipschitz spaces: the former are concerned with cumulative path behaviour while the latter are more sensitive to extremes in local variation. We also consider sample paths indexed by the half line, where the usual projective limit topologies are not strong enough for many applications, particularly in queueing theory.

Stronger topologies for sample path large deviations in Euclidean space

Neil O'Connell, BRIMS, Hewlett-Packard Labs, Bristol

BRIMS Technical Report HPL-BRIMS-96-005

Abstract

In this paper we present sufficient conditions for sample path large deviation principles to be extended to finer topologies. We consider extensions of the uniform topology by Orlicz functionals and we consider Lipschitz spaces: the former are concerned with cumulative path behaviour while the latter are more sensitive to extremes in local variation. We also consider sample paths indexed by the half line, where the usual projective limit topologies are not strong enough for many applications, particularly in queueing theory.

1 Introduction

Let $\mathcal{Z}$ be a topological space. A good rate function is a lower semicontinuous function $I : \mathcal{Z} \to \mathbb{R}_+$, with compact level sets $\{z : I(z) \leq \alpha\}$. A sequence of random variables Z_n, taking values in $\mathcal{Z}$, is said to satisfy the large deviation principle (LDP) with rate function I if for all closed $B \subset \mathcal{Z}$,

$$\limsup_n \frac{1}{n} \log P(Z_n \in B) \leq -\inf_{z \in B} I(z),$$

and for all open $G \subset \mathcal{Z}$,

$$\liminf_n \frac{1}{n} \log P(Z_n \in G) \geq -\inf_{z \in B} I(z).$$

Let X_k be a sequence of random variables in $\mathbb{R}^d$. For each $0 \leq t \leq 1$ set

$$S_n(t) = \frac{1}{n} \sum_{k=1}^{[nt]} X_k.$$

Denote by $L_\infty[0,1]$ the space of (equivalence classes modulo equality *a.e.* of) bounded measurable functions on $[0,1]$, equipped with the uniform topology, by $\mathcal{C}[0,1]$ the subspace of continuous functions, and by $\mathcal{A}[0,1]$ the subspace of absolutely continuous functions ϕ on $[0,1]$ with $\phi(0) = 0$.

It is a classical result, originally due to Mogulskii [14], that, if the (X_k) are *i.i.d.* and

$$\Lambda(\theta) = \log E e^{\theta \cdot X_1} < \infty$$

for all $\theta \in \mathbb{R}^d$, then the sequence S_n satisfies the LDP in $L^d_\infty[0,1]$ with good rate function given by

$$I(\phi) = \begin{cases} \int_0^1 \Lambda^*(\dot{\phi}) ds & \phi \in \mathcal{A}^d[0,1], \\ \infty & \text{otherwise} \end{cases} \tag{1}$$

where Λ^* is the convex dual of Λ:

$$\Lambda^*(x) = \sup_\theta [x \cdot \theta - \Lambda(\theta)].$$

The assumption that the moment generating function is finite has since been relaxed by Pinelis [18]; more recently, Dembo and Zajic [7] have relaxed the assumption of independence, replacing it with a variety of mixing and tightness hypotheses.

We find it convenient to consider, instead of S_n, its polygonal approximation:

$$\tilde{S}_n(t) = S_n(t) + \left(t - \frac{[nt]}{n}\right)\left(S_n\left(\frac{[nt]+1}{n}\right) - S_n\left(\frac{[nt]}{n}\right)\right). \qquad (2)$$

Note that $\tilde{S}_n$ is continuous, and carries the same information as S_n. The LDP for $\tilde{S}_n$ can typically be shown to hold under milder hypotheses, and in the Polish space $C^d[0,1]$ (equipped with the uniform topology).

In this paper we present sufficient conditions for such an LDP to be extended to finer topologies. In Section 3 we consider extensions of the uniform topology by Orlicz functionals and in Section 4 we consider Lipschitz spaces; the former are concerned with cumulative path behaviour while the latter are more sensitive to extremes in local variation. We also consider sample paths indexed by the half line, where the usual projective limit topologies are not strong enough for many applications, particularly in queueing theory. This extends work of Dobrushin and Pechersky [6].

2 Some facts about Orlicz spaces

What follows is mostly a summary of the relevant details from [1, Chapter 8]. A function $C : \mathbb{R}_+ \to \mathbb{R}_+$ is called an *N-function* if it is convex and satisfies

$$\lim_{x \to 0^+} C(x)/x = 0, \qquad \lim_{x \to \infty} C(x)/x = \infty.$$

Examples of N-functions are

$$C(x) = x^p, \quad (1 < p < \infty),$$

$$C(x) = e^x - x - 1,$$

$$C(x) = (1+x)\log(1+x) - x,$$

or

$$C_a(x) = C(x-a)1_{x \geq a}$$

for any N-function C. We will write C^* for the convex dual of C.

The *Orlicz class* K_C is the set of all (equivalence classes modulo equality *a.e.* of) measurable functions $f : [0,1] \to \mathbb{R}$ satisfying

$$\int_0^1 C(|f(s)|)ds < \infty;$$

the *Orlicz space* L_C is defined to be the linear hull of K_C, and is a Banach space when equipped with the *Luxemburg norm*

$$\|f\|_C = \inf\left\{k > 0 : \int_0^1 C\left(|f(s)|/k\right) ds \le 1\right\}.$$

We have the following generalisation of Hölder's inequality:

$$\left|\int_0^1 f(s)g(s)ds\right| \le 2\|f\|_C\|g\|_{C^*}. \tag{3}$$

An N-function C is said to satisfy a Δ_2-*condition near infinity* if there exists a positive constant $k > 0$ such that for all x sufficiently large,

$$C(2x) \le kC(x).$$

This is a useful condition, because $L_C = K_C$ if, and only if, C satisfies a Δ_2-condition near infinity [1, Lemma 8.8]; furthermore, L_C is reflexive, with dual L_{C^*}, if, and only if, both C and C^* satisfy a Δ_2-condition near infinity [1, Theorem 8.19]. Note that if L_C is reflexive, then it is weakly complete and the unit ball is weakly compact.

We will now record a lemma that will be useful for interpreting the results of the next section. A proof is given in the Appendix.

Lemma 1 *Suppose that C^* satisfies a Δ_2-condition near infinity. A sequence f_n is weakly convergent to a function f in L_C if, and only if,*

(i) $\|f_n\|_C$ is bounded, and

(ii) for each $t \in [0,1]$, $\int_0^t f_n(s)ds \to \int_0^t f(s)ds$.

Note that all of the above remarks apply to Orlicz spaces of functions on any finite interval $[0,t]$, which we denote by $L_B[0,t]$.

3 The LDP in a topology extended by Orlicz functionals

(H1) For each $\theta \in \mathbb{R}^d$, the limit

$$\Lambda(\theta) = \lim_{n\to\infty} \frac{1}{n} \log E e^{n\theta \cdot S_n(1)}, \tag{4}$$

exists as an extended real number. The sequence $\tilde{S}_n$ satisfies the LDP in $C^d[0,1]$ with good rate function given by

$$I(\phi) = \begin{cases} \int_0^1 \Lambda^*(\dot{\phi})ds & \phi \in \mathcal{A}^d[0,1], \\ \infty & \text{otherwise} \end{cases}$$

where Λ^* is the convex dual of Λ.

For each $j = 1, \ldots, d$ and $\lambda \in \mathbb{R}$, set

$$\Lambda_j(\lambda) = \lim_{n\to\infty} \frac{1}{n} \log E e^{n\lambda S_n^j(1)}.$$

Note that the existence of Λ_j follows from the existence of Λ, and can be obtained by evaluating Λ at multiples of the j^{th} standard basis vector.

(H2) (i) For each $j = 1, \ldots, d$, there exists an N-function B_j with

$$\limsup_{|x|\to\infty} \frac{B_j(|x|)}{\Lambda_j^*(x)} < \infty,$$

and both B_j and its convex dual satisfy a Δ_2-condition near infinity.

(ii) For some $\delta > 0$,

$$\limsup_{n\to\infty} \frac{1}{n} \log E \exp\left(\delta \sum_{k=1}^n B_j(|X_k^j|)\right) < \infty,$$

for each $j = 1, \ldots, d$.

The hypothesis (H2) is guaranteed in many situations. For example, if the (X_k) are bounded, it holds. If the (X_k) are *i.i.d.*, then (i) implies (ii) (this follows from [8, Lemma 5.1.14]). The hypothesis (i) is certainly guaranteed if the effective domain of Λ^* is finite. The hypothesis (ii), although it is essentially a mixing condition, appears to have no direct connection with the mixing conditions introduced in [7].

Let (B_j) be the N-functions of (H2), and write B for the $\mathbb{R}^d$-valued function with coordinates (B_j). Recall that if ϕ is absolutely continuous, then $\dot{\phi}$ exists almost everywhere. Define a space

$$\mathcal{X}_B = \{\phi \in \mathcal{A}^d[0,1] : \dot{\phi}^j \in L_{B_j},\ j = 1, \ldots, d\}.$$

We will consider a variety of topologies on $\mathcal{X}_B$: the uniform topology, which we will denote by τ_u; the topology inherited from the product space $\prod_j L_{B_j}$, which we will denote by τ_B; the corresponding weak topology, τ_B^w; and the smallest topology containing both τ_u and τ_B^w, which we will denote by σ_B. The topology σ_B is the smallest extension of the uniform topology for which the functions

$$\phi \mapsto \int_0^1 \dot{\phi}^j(s)g(s)ds$$

are continuous for all $g \in L_{B_j}$, and for each $j = 1, \ldots, d$. By Lemma 1, a sequence ϕ_n converges in this topology if, and only if, it converges uniformly and the sequences $\|\dot{\phi}_n^j\|_{B_j}$ are bounded.

Theorem 1 *If (H1) and (H2) are satisfied then $\tilde{S}_n$ satisfies the LDP in $(\mathcal{X}_B, \sigma_B)$, with good rate function given by*

$$I(\phi) = \int_0^1 \Lambda^*(\dot{\phi})ds.$$

Proof. It follows from (H1) and [8, Lemma 4.1.5] that $\tilde{S}_n$ satisfies the LDP in this space when equipped with the relative (uniform) topology; here we have also used (H2) and the fact that $L_B = K_B$ to infer that $\mathcal{D}_I \subset \mathcal{X}_B^d$. To strengthen this to the space $(\mathcal{X}_B, \sigma_B)$ we appeal to the 'inverse contraction principle' [8, Theorem 4.2.4, Corollary 4.2.6], by which it suffices to prove that the sequence $\tilde{S}_n$ is exponentially tight in this space.

It is convenient to consider the natural imbedding of the weak topology τ_B^w in the space $C^d[0,1]$. This is complete, by (H2). Goodness of the rate function in (H1) implies exponential tightness in the space $C^d[0,1]$, when equipped with the uniform topology, since this is a Polish space (see, for example, [8, Exercise 4.1.10]). Thus, by Lemma 6, we need only check that $\tilde{S}_n$ is exponentially tight in $(\mathcal{X}_B^d, \tau_B^w)$.

For each $\alpha > 0$, set

$$K_\alpha = \{\phi \in \mathcal{X}_B : \max_j \|\dot{\phi}^j\|_{B_j} \le \alpha\}.$$

By the hypothesis (H2), the spaces L_{B_j} are reflexive and so K_α is compact in $(\mathcal{X}_B^d, \tau_B^w)$. For each $\alpha > 0$, set

$$C_\alpha = \{\phi \in \mathcal{A}^d[0,1] : \max_j \int_0^1 B_j(|\dot{\phi}^j|)ds \le \alpha\}.$$

Lemma 2 *For α sufficiently large, $C_\alpha \subset K_\alpha$.*

Proof. For α sufficiently large we have, by convexity of the B_j, that

$$\begin{aligned} \max_j \int_0^1 B_j(|\dot{\phi}^j|)ds \le \alpha \quad &\Rightarrow \quad \max_j \alpha \int_0^1 B_j(|\dot{\phi}^j|/\alpha)ds \le \alpha \\ &\Rightarrow \quad \max_j \int_0^1 B_j(|\dot{\phi}^j|/\alpha)ds \le 1 \\ &\Rightarrow \quad \max_j \|\dot{\phi}^j\|_{B_j} \le \alpha. \end{aligned}$$

□

The exponential tightness of $\tilde{S}_n$ in $(\mathcal{X}_B, \tau_B^w)$, and hence the statement of the theorem, is thus established by the following lemma.

Lemma 3 *If (H2) is satisfied, then*

$$\lim_{\alpha\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log P(\tilde{S}_n \in C_\alpha^c) = -\infty.$$

Proof. By Chebyshev's inequality we have, for $\alpha, \delta > 0$,

$$\begin{aligned} P(\tilde{S}_n \in C_\alpha^c) &= P\left(\max_j \frac{1}{n}\sum_{k=1}^n B_j(|X_k^j|) > \alpha\right) \\ &\le e^{-\delta\alpha n} \max_j E \exp\left(\delta \sum_{k=1}^n B_j(|X_k^j|\right). \end{aligned}$$

The result is obtained by choosing $\delta > 0$ as in (H2), considering the normalised logarithmic limit as $n \to \infty$, and then letting $\alpha \to \infty$. □

4 The LDP in Lipschitz spaces

For $\alpha \ge 0$ we denote by $\mathrm{Lip}_\alpha[0,1]$ the space of α-Hölder continuous functions on $[0,1]$, equipped with the norm

$$\|\phi\|_\alpha = \sup_{s\neq t} \frac{|\phi(t) - \phi(s)|}{|t-s|^\alpha}. \tag{5}$$

Set $R_j(x) = \Lambda^*(x) \wedge \Lambda^*(-x)$. We will begin by recording some hypotheses. Note that (H5) follows automatically from (H1) if the (X_k) are stationary.

(H1a) (H1) holds with Λ lower semicontinuous and differentiable at the origin.

(H4) For each j, there exists $0 \le \beta_j \le 1$ such that

$$\liminf_{r\to\infty} \inf_{0<\tau<1} \tau R_j(r\tau^{\beta_j - 1}) = +\infty.$$

(H5) For each $\epsilon > 0$ and $\theta \in \mathbb{R}^d$, there exists $C > 0$, such that

$$E \exp(\theta \cdot [S_n(t) - S_n(s)]) \le C \exp(n(t-s)[\Lambda(\theta) + \epsilon]),$$

for all $0 \le s < t \le 1$.

Theorem 2 *If (H1a), (H4) and (H5) are satisfied and $0 < \alpha_j < \beta_j$ for each j, then the sequence $\tilde{S}_n$ satisfies the LDP in $\prod_j Lip_{\alpha_j}[0,1]$ with good rate function.*

Proof. First note that (H4) implies

$$\mathcal{D}_I \subset \prod_j \mathrm{Lip}_{\beta_j}[0,1] \subset \prod_j \mathrm{Lip}_{\alpha_j}[0,1].$$

Indeed, if $\|\phi^j\|_{\beta_j} \ge r$, for some j, then

$$\begin{aligned} I(\phi) &= \int_0^1 \Lambda^*(\dot{\phi}) ds \\ &\ge \int_0^1 \Lambda_j^*(\dot{\phi}^j) ds \\ &\ge \inf_{0<\tau<1} \tau R_j(r\tau^{\beta_j - 1}). \end{aligned}$$

We also have, trivially, that

$$P\left(\tilde{S}_n \in \prod_j \mathrm{Lip}_{\alpha_j}[0,1]\right) = 1,$$

for all n. Therefore, it suffices to prove that $\tilde{S}_n$ is exponentially tight in the space $\prod_j \mathrm{Lip}_{\alpha_j}[0,1]$. To do this, we consider the sets

$$K(r) = \bigcap_j \{\phi : \|\phi^j\|_{\beta_j} \le r\}.$$

These are compact in $\prod_j \mathrm{Lip}_{\alpha_j}[0,1]$ (*c.f.* [13, Corollary 3.3]). By Chebychev's inequality and (H5) we have, for each $\epsilon > 0$ and for each pair $\theta_1, \theta_2 \in \mathbb{R}$,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log P(\|\tilde{S}_n^j\|_{\beta_j} \geq r) \\
&\leq \limsup_{n\to\infty} \sup_{0<s<t<1} \frac{1}{n} \log P(|S_n^j(t) - S_n^j(s)| > r(t-s)_j^{\beta}) \\
&\leq \left[\sup_{0<s<t<1} [(t-s)\Lambda_j(\theta_1) - r(t-s)^{\beta_j}\theta_1] + \epsilon\right] \\
&\quad \vee \left[\sup_{0<s<t<1} [(t-s)\Lambda_j(\theta_2) + r(t-s)^{\beta_j}\theta_2] + \epsilon\right].
\end{aligned}$$

Since this holds for all ϵ we can let $\epsilon \to 0$ to get

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log P(\|\tilde{S}_n^j\|_{\beta_j} \geq r) \\
&\leq \sup_{0<\tau<1} [\tau\Lambda_j(\theta_1) - r\tau^{\beta_j}\theta_1] \vee \sup_{0<\tau<1} [\tau\Lambda_j(\theta_2) + r\tau^{\beta_j}\theta_2].
\end{aligned}$$

Next we take the infimum over $\theta_1 \geq 0$ and $\theta_2 \leq 0$. To justify the interchange of inf and sup, note that wherever $\theta_1 \geq 0$ and $\Lambda_j(\theta_1)$ is finite,

$$(\tau, \theta_1) \mapsto \tau\Lambda_j(\theta_1) - r\tau^{\beta_j}\theta_1$$

is a closed, proper, concave-convex function (see, for example, [20, Corollary 37.3.1]). Here we have used the fact that Λ is lower semicontinuous and proper, by (H1a) (see, for example, [8, Lemma 2.3.9]). A similar argument applies to the second term when $\theta_2 \leq 0$, so we have

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log P(\|\tilde{S}_n^j\|_{\beta_j} \geq r) \\
&\leq \sup_{0<\tau<1} \inf_{\theta_1\geq 0} [\tau\Lambda_j(\theta_1) - r\tau^{\beta_j}\theta_1] \vee \sup_{0<\tau<1} \inf_{\theta_2\leq 0} [\tau\Lambda_j(\theta_2) + r\tau^{\beta_j}\theta_2].
\end{aligned}$$

Finally, we use that fact (see, for example, [8, Lemma 2.2.5]) that for $x > |\Lambda_j'(0)|$,

$$\Lambda_j^*(x) = \sup_{\theta\geq 0}[\theta x - \Lambda_j(\theta)]$$

and

$$\Lambda_j^*(-x) = \sup_{\theta\leq 0}[\theta x - \Lambda_j(\theta)],$$

to deduce that for r sufficiently large,

$$\limsup_{n\to\infty} \frac{1}{n} \log P(\|\tilde{S}_n^j\|_{\beta_j} \geq r) \leq - \inf_{0<\tau<1} \tau R_j(r\tau^{\beta_j - 1}).$$

Exponential tightness now follows from (H4). □

Note that if $\Lambda^*(x) = |x|^2$, then (H4) is satisfied with $\beta_j = 1/2$ for each j. This provides a generalisation of Schilder's theorem and was presented as an exercise in [8, Exercise 5.2.14]. If the effective domain of Λ^* is finite, then (H4) holds with $\beta_j = 1$ for each j.

5 The LDP for sample paths indexed by $\mathbb{R}_+$

Denote by $\mathcal{C}(\mathbb{R}_+)$ the space of continuous functions on $\mathbb{R}_+$ and by $\mathcal{A}(\mathbb{R}_+)$ the subspace of absolutely continuous functions starting at zero. Consider the sequence of partial sums processes $\tilde{S}_n$ defined by (2) on the entire half line. Using the Dawson-Gärtner theorem for projective limits, an immediate consequence of (H1) is that the sequence $\tilde{S}_n$ satisfies the LDP in $\mathcal{C}^d(\mathbb{R}_+)$ when equipped with the topology of uniform convergence on compact intervals, with good rate function given by

$$I(\phi) = \begin{cases} \int_0^\infty \Lambda^*(\dot{\phi})ds & \phi \in \mathcal{A}^d(\mathbb{R}_+) \\ \infty & \text{otherwise.} \end{cases} \tag{6}$$

However, as is pointed out by Dobrushin and Pechersky in [6], this topology is not strong enough for many applications. For example, a typical quantity of interest in queueing theory is the steady-state queue-length at a deterministic buffer with inputs given by a real-valued stationary sequence (X_k): this can be represented by

$$Q = n \sup_{t\geq 0} [\tilde{S}_n(t) - t].$$

To apply the contraction principle, and hence deduce tail asymptotics for Q, one requires that the mapping

$$\phi \mapsto \sup_{t\geq 0} [\phi(t) - t] \tag{7}$$

is continuous with respect to the topology on path space for which the LDP is assumed to hold. Dobrushin and Pechersky [6] introduce a finer topology (a guage topology) for which the restriction of above mapping to a subspace of non-decreasing paths is continuous, and prove the LDP in this topology for a class of Markov jump processes. In this section we will demonstrate that the hypothesis (H1), plus differentiability of Λ at the origin, is sufficient to strengthen the LDP to a finer topology for which mappings such as (7) are continuous (without the non-decreasing path restriction). The hypothesis (H2), in addition, yields the LDP in a topology which is finer again.

We consider the set of paths

$$\mathcal{Y} = \bigcap_j \left\{ \phi \in C^d(\mathbb{R}_+) : \lim_{t\to\infty} \frac{\phi^j(t)}{1+t} \text{ exists } \right\},$$

and equip $\mathcal{Y}$ with the norm

$$\|\phi\|_u = \sup_j \sup_t \left| \frac{\phi^j(t)}{1+t} \right|.$$

Note that $\mathcal{Y}$ can be identified with the Polish space $C^d(\mathbb{R}_+^*)$ of continuous functions on the extended (and compactified) real line, equipped with the supremum norm, via the bijective mapping $\phi(t) \mapsto \phi(t)/(1+t)$. In particular, $\mathcal{Y}$ is a Polish space.

Although this topology is quite different from the gauge topology introduced by Dobrushin and Pechersky [6], conceptually it is quite similar: the idea is to get some kind of uniform control over the sample average. We have also used some ideas from their paper in the proof of Theorem 3 below, in order to construct compact sets that support most of the measure.

Theorem 3 *If (H1) holds, and Λ is differentiable at the origin, then $\tilde{S}_n$ satisfies the LDP in $\mathcal{Y}$ with good rate function given by (6).*

Theorem 3 provides a new tool for looking at large deviations for queueing systems in equilibrium. Equilibrium systems have generally been treated on a case-by-case basis, and much work and/or additional hypotheses are required to prove large deviation principles (see, for example, Chang and Zajic [4] or Ramanan and Dupuis [19]). To illustrate this, consider the function f defined by (7) in the case $d = 1$. Recall that $f(\tilde{S}_n)$ is equal in distribution to the normalised queue length at a single-server queue. If $\Lambda'(0) = \mu$, say, then a corollary of Theorem 3 is that the LDP holds in the subspace

$$\mathcal{Y}^\mu = \left\{ \phi \in \mathcal{Y} : \lim_{t\to\infty} \frac{\phi(t)}{1+t} = \mu \right\}.$$

If $\mu < 1$, then the restriction of f to $\mathcal{Y}^\mu$ is finite and continuous. To see this, observe that if $\|\phi - \phi'\|_u < \epsilon$, then there exists a t_0, *independent of* ϕ, such that $|f(\phi) - f(\phi')| < 2t_0\epsilon$. We can therefore apply the contraction principle and Jensen's inequality to get that the sequence $Q/n = f(\tilde{S}_n)$ satisfies the LDP in $\mathbb{R}_+$ with rate function given by

$$\begin{aligned} J(q) &= \inf\left\{\int_0^\infty \Lambda^*(\dot{\phi})ds : \sup_{t>0}[\phi(t) - ct] = q\right\} \\ &= \inf_{\tau>0}\inf\left\{\int_0^\tau \Lambda^*(\dot{\phi})ds : \phi(\tau) - c\tau = q\right\} \\ &= \inf_{\tau>0} \tau\Lambda^*(c + q/\tau). \end{aligned}$$

This fact has previously been demonstrated by several authors [3, 9, 10, 12], under similar conditions. The *i.i.d.* case is due to Cramér [5] and Borovkov [2]. The advantage of the above approach is that the existence of an LDP is established by continuity which, using the above topology, is quite accessible, and the rate function is easier to compute. For more complicated examples of this approach, see [15, 16, 17].

Proof of Theorem 3. By considering $\tilde{S}_n(t) - t\nabla\Lambda(0)$ we can, without loss of generality, assume that $\nabla\Lambda(0) = 0$. We first show that $\mathcal{D}_I \subset \mathcal{Y}$ and $P(\tilde{S}_n \in \mathcal{Y}) = 1$, for all n. By the convexity of Λ^*, and Jensen's inequality,

$$t\Lambda^*(\phi(t)/t) \leq I(\phi).$$

Since this holds for all t, we must have $\phi(t)/t \to 0$ as $t \to \infty$ ($\nabla\Lambda(0) = 0$ implies that Λ^* has a unique zero at the origin). By hypothesis we can choose, for each j, $\theta > 0$ such that $\Lambda_j(\theta)$ and $\Lambda_j(-\theta)$ are finite; if we let

$$\epsilon_+(n) = \left|\frac{1}{n}\log Ee^{n\theta S_n^j(1)} - \Lambda_j(\theta)\right|$$

and

$$\epsilon_-(n) = \left|\frac{1}{n}\log Ee^{-n\theta S_n^j(1)} - \Lambda_j(-\theta)\right|,$$

then $\epsilon_+(n) \vee \epsilon_-(n) \to 0$, as $n \to \infty$. Thus, for each $\delta > 0$,

$$\begin{aligned} &P\left\{\left|\frac{\tilde{S}_n^j(t)}{1+t}\right| > \delta, \text{ for some } t > t_0\right\} \\ &\leq \sum_{k>nt_0} e^{-\delta k\theta + \Lambda(\theta) + \epsilon_+(k)} + \sum_{k>nt_0} e^{-\delta k\theta + \Lambda(-\theta) + \epsilon_-(k)}; \end{aligned}$$

letting $t_0 \to \infty$ we see that $\tilde{S}_n^j(t)/(1+t) \to 0$, almost surely, as $t \to \infty$. We have thus shown that $\mathcal{D}_I \subset \mathcal{Y}$ and $P(\tilde{S}_n \in \mathcal{Y}) = 1$. Now by (H1), the Dawson-Gärtner theorem for projective limits, and [8, Lemma 4.1.5], we have that $\tilde{S}_n$ satisfies the LDP in $\mathcal{Y}$ when equipped with the topology of uniform convergence on compact intervals. To strengthen this to the topology induced by the norm $\|\cdot\|_u$, we appeal again to the inverse contraction principle, by which it suffices to prove exponential tightness in the space $(\mathcal{Y}, \|\cdot\|_u)$.

For each t, denote by $C^d[0,t]$ the projection of $C^d(\mathbb{R}_+)$ onto the interval $[0,t]$, equipped with the uniform topology, and by $\phi[0,t]$, for $\phi \in C^d(\mathbb{R}_+)$, the restriction of ϕ to the interval $[0,t]$. Goodness of the rate function in (H1) implies that the sequence $\tilde{S}_n[0,1]$ is exponentially tight in the uniform topology on $\mathcal{A}^d[0,1]$. In other words, for each $\alpha > 0$, there exists a compact set K_α in $\mathcal{A}^d[0,1]$ such that

$$\limsup_n \frac{1}{n} \log P(\tilde{S}_n[0,1] \notin K_\alpha) \leq -\alpha.$$

It follows that for each $t > 0$,

$$K_\alpha(t) := \{\phi \in C^d[0,t] : \ \{s \mapsto \phi(st)\} \in K_\alpha\}$$

is compact in $C^d[0,t]$, and for each $0 < \epsilon < \alpha$,

$$\begin{aligned} \limsup_n \frac{1}{n} \log P \bigcup_{t>1} \{\tilde{S}_n[0,t] \notin K_\alpha(t)\} &\leq \limsup_n \frac{1}{n} \log P \bigcup_{t>1} \{\tilde{S}_{nt}[0,1] \notin K_\alpha(1)\} \\ &\leq \limsup_n \frac{1}{n} \log \sum_{k>n} e^{-(\alpha-\epsilon)k} \\ &\leq -\alpha + \epsilon. \end{aligned}$$

Since ϵ is arbitrary we have, for each $\alpha > 0$,

$$\limsup_n \frac{1}{n} \log P \bigcup_{t>1} \{\tilde{S}_n[0,t] \notin K_\alpha(t)\} \leq -\alpha. \tag{8}$$

For $\alpha, t > 0$, set

$$d_\alpha(t) = \begin{cases} \alpha^2 & t \leq \alpha^2 \\ t^{-1/2} & t > \alpha^2 \end{cases}$$

and consider the sets

$$D_\alpha = \bigcap_j \left\{ \phi \in \mathcal{Y} : \ \left| \frac{\phi^j(t)}{1+t} \right| \leq d_\alpha(t), \text{ for all } t, \ \phi[0,t] \in K_\alpha(t) \text{ for all } t > 1 \right\}.$$

The exponential tightness of $\tilde{S}_n$ in $(\mathcal{Y}, \|\cdot\|_u)$ will be established by the following two lemmas.

Lemma 4 *For each $\alpha > 0$, D_α is compact in $(\mathcal{Y}, \|\cdot\|_u)$.*

Proof. Let ϕ_n be a sequence in D_α. By Tychonoff's theorem, the set $\cap_{t>1} K_\alpha(t)$ is compact in $\mathcal{Y}$ when equipped with the topology of uniform convergence on compact intervals, so there exists a subsequence $n(k)$ such that ϕ converges to some $\phi \in \cap_{t>1} K_\alpha(t)$ in this topology. It follows that, for each $T > 0$, and for each j,

$$\lim_{k\to\infty} \sup_{t \le T} \left| \frac{\phi^j_{n(k)}(t)}{1+t} - \frac{\phi^j(t)}{1+t} \right| = 0.$$

Note that this implies, for each t and j,

$$\left| \frac{\phi^j(t)}{1+t} \right| \le d_\alpha(t),$$

and so $\phi \in D_\alpha$. Now for each $\epsilon > 0$ (sufficiently small), we have for k sufficiently large,

$$\begin{aligned} \|\phi_{n(k)} - \phi\|_u &\le \sup_{t \le 1/\epsilon^2} \left| \frac{\phi^j_{n(k)}(t)}{1+t} - \frac{\phi^j(t)}{1+t} \right| + \sup_{t > 1/\epsilon^2} \left| \frac{\phi^j_{n(k)}(t)}{1+t} - \frac{\phi^j(t)}{1+t} \right| \\ &\le \epsilon + 2d_\alpha(1/\epsilon^2) = 3\epsilon. \end{aligned}$$

The set D_α is therefore sequentially compact, and hence compact, in $(\mathcal{Y}, \|\cdot\|_u)$. □

Lemma 5 *If (H1) is satisfied, then*

$$\lim_{\alpha\to\infty} \limsup_n \frac{1}{n} \log P(\tilde{S}_n \notin D_\alpha) = -\infty.$$

Proof. First we have, by the contraction principle,

$$\limsup_n P \bigcup_{t \le \alpha^2} \{|\tilde{S}_n(t)| > \alpha^2(1+t)\} \le - \inf_{0<\tau<\alpha^2} \tau R_j(\alpha^2/\tau) \le -\alpha^2 R_j(1). \tag{9}$$

Here we have used that fact that $\tau\Lambda^*(\alpha^2/\tau)$ and $\tau\Lambda^*(-\alpha^2/\tau)$ are both non-increasing functions of τ, which can be checked using Jensen's inequality. We also have, for each j and some $\theta > 0$,

$$\begin{aligned} P \bigcup_{t>\alpha^2} \{|\bar{S}_n^j(t)| > (1+t)d_\alpha(t)\} &\leq P \bigcup_{i=0}^{n-1} \bigcup_{k=[\alpha^2]}^{\infty} \{|\bar{S}_n^j(k+i/n)| > (1+k)d_\alpha(k)\} \\ &\leq n \sum_{k=[\alpha^2]}^{\infty} C(\theta)e^{-\theta nkd_\alpha(k)} \\ &\leq nC(\theta)De^{-\theta n\sqrt{\alpha^2-1}/2}. \end{aligned}$$

Here we have used (H1), Chebyshev's inequality, and the inequality

$$\sum_{k\geq k_0} e^{-\rho\sqrt{k}} \leq De^{-\rho\sqrt{k_0-1}/2}.$$

It follows that

$$\limsup_n \frac{1}{n} \log P \bigcup_{t>\alpha^2} \{|\bar{S}_n^j(t)| > (1+t)d_\alpha(t)\} \leq -\theta\sqrt{\alpha^2-1}/2. \qquad (10)$$

The statement can now be obtained from (8), (9) and (10), via the principle of the largest term. □

This concludes the proof of the theorem. □

We will now conclude this section with the observation that the hypothesis (H2), in addition, yields the LDP with respect to an even finer topology. Consider the space

$$\mathcal{Y}_B = \bigcap_j \{\phi \in \mathcal{Y} : \dot{\phi}^j[0,t] \in L_{B_j}[0,t] \text{ for all } t > 0\},$$

where B_j are the N-functions of hypothesis (H2). Denote by $\tau_B^w(t)$ the product of the weak topologies on $\prod_j L_{B_j}[0,t]$, and by $\tau_B^w(\infty)$ the projective limit, as $t \to \infty$, of the $\tau_B^w(t)$. The restriction of $\tau_B^w(\infty)$ to $\mathcal{Y}_B$ is the weakest topology for which the mappings $\phi \mapsto \int_0^t g(s)\dot{\phi}(s)ds$ are continuous, for all $t > 0$ and $g \in \prod_j L_{B_j}[0,t]$. Now write ω_B for the smallest topology on $\mathcal{Y}_B$ containing the topology of the norm $\|\cdot\|_u$ and the restriction of $\tau_B^w(\infty)$.

Theorem 4 *If (H1) and (H2) are satisfied, and Λ is differentiable at the origin, then $\bar{S}_n$ satisfies the LDP in $(\mathcal{Y}_B, \omega_B)$ with good rate function given by (6).*

Proof. It suffices to prove that $\tilde{S}_n$ is exponentially tight in $(\mathcal{Y}_B, \tau_B^w(\infty))$. This follows from Lemma 6, as in the proof of Theorem 1: $\tau_B^w(\infty)$ can be thought of as a projective limit of weak topologies, each of which is complete by (H2), and so the natural embedding of $\tau_B^w(\infty)$ on $\mathcal{Y}$ is also complete; by the previous theorem, we have exponential tightness in $\mathcal{Y}$, which is Polish.

The sets

$$K_\alpha = \bigcap_j \{\phi \in \mathcal{Y}_B : \sup_t \frac{1}{1+t} \|\dot{\phi}^j[0,t]\|_{B_j} \leq \alpha\}$$

are compact in $(\mathcal{Y}_B, \tau_B^w(\infty))$, by Tychonoff's theorem. It follows (*c.f.* Lemma 2) that, for α sufficiently large, the sets

$$C_\alpha = \bigcap_j \{\phi \in \mathcal{Y}_B : \sup_t \frac{1}{1+t} \int_0^t B_j(|\dot{\phi}^j(s)|) ds \leq \alpha\}$$

are precompact. Now, by (H2),

$$\begin{aligned} P(\tilde{S}_n \notin C_\alpha) &= P \bigcup_t \left\{ \sum_{k=1}^{[nt]} B_j(|X_k^j|) > \alpha n(1+t) \right\} \\ &\leq \sum_{l=1}^{\infty} P \left\{ \sum_{k=1}^{l} B_j(|X_k^j|) > \alpha(n+l) \right\} \\ &\leq e^{-\delta \alpha n} \sum_{l=1}^{\infty} e^{-\delta \alpha l + Cl}, \end{aligned}$$

for some constants $\delta, C > 0$. For α sufficiently large, the last summation is finite and bounded, and the result follows. □

Acknowledgements. I would like to thank Amir Dembo, John Lewis, George O'Brien and Balaji Prabhakar for helpful discussions and comments.

Appendix

Lemma 6 *If are τ_1 and τ_2 are both complete metric topologies on a space $\mathcal{Z}$, and the sequence Z_n is exponentially tight with respect to both topologies, then it is exponentially tight with respect to $\tau_1 \vee \tau_2$ (the smallest topology containing τ_1 and τ_2).*

Proof. Let d_1 and d_2 be a pair of complete metrics that respectively generate the topologies τ_1 and τ_2. By hypothesis, for every $\alpha > 0$, there exist τ_i-compact sets K_α^i for which

$$\limsup_n \frac{1}{n} \log P(Z_n \notin K_\alpha^i) \leq -\alpha.$$

It follows from the principle of the largest term that

$$\limsup_n \frac{1}{n} \log P(Z_n \notin K_\alpha^1 \cap K_\alpha^2) \leq -\alpha.$$

Recall that a set in a complete metric topology is precompact if, and only if, it is totally bounded. The sets $K_\alpha^1 \cap K_\alpha^2$ are thus totally bounded in both metrics, and hence in the metric $d_1 \vee d_2$, which generates the topology $\tau_1 \vee \tau_2$. The statement follows. □

Proof of Lemma 1. Suppose f_n converges weakly to f in $\bar{L}_C$. In other words, for every $g \in L_{C^*}$,

$$\int_0^1 f_n(s)g(s)ds \to \int_0^1 f(s)g(s)ds. \tag{11}$$

Then (i) follows from general principles (see, for example, [11, Theorem 4.10.7]) and (ii) follows from (11).

Now suppose that (i) and (ii) hold. To deduce that

$$\int_0^1 f_n(s)g(s)ds \to \int_0^1 f(s)g(s)ds,$$

for every $g \in L_{C^*}$, we need the following lemma.

Lemma 7 *If A is an N-function satisfying a Δ_2-condition near infinity, then any $h \in L_A$ can be written as an almost everywhere limit of step functions h_n with $\|h_n - h\|_A \to 0$.*

Proof. Without loss of generality we can assume that h is non-negative. (For general h simply split it into its positive and negative parts.) Then h can be written as a monotone increasing sequence of simple functions (see, for example, [11, Theorem 2.2.5]), which therefore converge in the mean and [1], since A satisfies a Δ_2-condition near infinity, in L_A. Furthermore, each simple function can be written as an almost everywhere limit of step functions, which also converge in the mean, by bounded convergence, and hence in L_A. □

Now (ii) implies that

$$\left|\int f_n g - \int f g\right| \to 0,$$

for all step functions g. For general $g \in L_{C^*}$, write g as a limit of step functions g_m, as in the above lemma, so that $\|g - g_m\|_{C^*} \to 0$. Now we have, applying (3) for each m,

$$\begin{aligned}
&\limsup_n \left|\int f_n g - \int f g\right| \\
&= \limsup_n \left|\int f_n g - \int f_n g_m + \int f_n g_m - \int f g_m + \int f g_m - \int f g\right| \\
&\leq \limsup_n \left\{2\|f_n\|_C \|g - g_m\|_{C^*} + 2\|f\|_C \|g - g_m\|_{C^*} + \left|\int f_n g_m - \int f g_m\right|\right\} \\
&\leq 2(\alpha + \|f\|_C)\|g - g_m\|_{C^*}
\end{aligned}$$

where $\alpha = \sup_n \|f_n\|_C$, and since this holds for all m we can let m tend to infinity for the result. □

References

[1] Robert A. Adams. *Sobolev Spaces.* Academic Press, 1975.

[2] A.A. Borovkov. *Random Processes in Queueing Theory.* Springer-Verlag, 1976.

[3] Cheng-Shang Chang. Stability, queue length and delay of deterministic and stochastic queueing networks. *IEEE Trans. on Automatic Control* 39:913–931, 1994.

[4] C.-S. Chang and T. Zajic. Effective bandwidths of departure processes from queues with time varying capacities. INFOCOM, 1995.

[5] H. Cramér. On some questions connected with mathematical risk. Univ. Calif. Publications in Statistics, vol. 2, 99–125, 1954.

[6] R.L. Dobrushin and E.A. Pechersky. Large deviations for random processes with independent increments on infinite intervals. Preprint.

[7] Amir Dembo and Tim Zajic. Large deviations: from empirical mean and measure to partial sums process. *Stoch. Proc. Appl.* 57:191–224, 1995.

[8] Amir Dembo and Ofer Zeitouni. *Large Deviations Techniques and Applications.* Jones and Bartlett, 1993.

[9] G. de Veciana, C. Courcoubetis and J. Walrand. Decoupling bandwidths for networks: a decomposition approach to resource management. Memorandum No. UCB/ERL M93/50, University of California, 1993.

[10] N.G. Duffield and Neil O'Connell. Large deviations and overflow probabilities for the general single server queue, with applications. *Proc. Camb. Phil. Soc.* 118(1), 1995.

[11] Avner Friedman. *Foundations of Modern Analysis.* Dover, 1982.

[12] Peter W. Glynn and Ward Whitt. Logarithmic asymptotics for steady-state tail probabilities in a single-server queue. *J. Appl. Prob.*, to appear.

[13] J. A. Johnson. Banach spaces of Lipschitz functions and vestor valued Lipschitz functional. *Trans. Amer. Math. Soc.* 148: 147–169, 1970.

[14] A.A. Mogulskii. Large deviations for trajectories of multi dimensional random walks. *Th. Prob. Appl.* 21:300–315, 1976.

[15] Neil O'Connell. Queue lengths and departures at single-server resources. To appear in the *Proceedings of the RSS Workshop on Stochastic Networks*, Edinburgh, 1995.

[16] Neil O'Connell. Large deviations for departures from a shared buffer. Revision submitted to *J. Appl. Prob.*

[17] Neil O'Connell. Large deviations for queue lengths at a multi-buffered resource. Revision submitted to *J. Appl. Prob.*

[18] I.F. Pinelis. A problem on large deviations in the space of trajectories. *Th. Prob. Appl.* 26:69–84, 1981.

[19] Kavita Ramanan and Paul Dupuis. Large deviation properties of data streams that share a buffer. Technical Report LCDS 95-8, Division of Applied Mathematics, Brown University.

[20] R.T. Rockafeller. *Convex Analysis.* Princeton University Press, Princeton, New Jersey, 1970.

BRIMS, Hewlett-Packard Labs
Filton Road, Stoke Gifford
Bristol BS12 6QZ, UK.