Quantum Measurement Back-Reaction and Induced Topological Phases

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It is shown that a topological vector-potential (Berry phase) is induced by the act of measuring angular momentum in a direction defined by a reference particle. This vector potential appears as a consequence of the back-reaction due to the quantum measurement.

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Observables in an isolated system need to commute with any conserved quantity of the system. They need therefore to be defined, in classical mechanics as well as in quantum mechanics, as relative observables [1, 2]. Consider for example a measurement of the angular momentum, $\vec{L}_A$, of particle A. Since $\vec{L}_A$ generally does not commute with the conserved total angular momentum, $\vec{L}_{Total} = \vec{L}_A + \vec{L}_B + \cdots$, it can not be observed. In practice however, we always specify an axis $\hat{n}_B$, where B refers to another element of the system, with respect to which $\hat{n}_B \cdot \vec{L}_A$ is measured. The *relative* angular momentum, $\hat{n}_B \cdot \vec{L}_A$, indeed commutes with $\vec{L}_{Total}$. We notice that a measurement of $\hat{n}_B \cdot \vec{L}_A$, obviously requires interacting with both the system and the reference system and thus affects both. It is here that the quantum mechanical case differs considerably from the classical case. The uncertainty principle implies that unlike classical mechanics, the quantum mechanical back-reaction must always be sufficiently large. In the example eluded above, the measurement rotates the angular momentum components, which do not commute with $\hat{n}_B \cdot \vec{L}_A$, by an uncertain large angle.

In this letter we show that, the "strong" nature of the quantum mechanical back-reaction on the reference system, can in particular cases give rise to an effective topological vector-potential. This induced topological effect can be interpreted as a Berry phase, thus leading to a fundamental relation between quantum measurements and the Berry effect.

To begin with, let us consider a measurement of a half-integer spin $\vec{s}$. The role of the reference system is to provide us with a direction $\hat{n} = \vec{r}/|\vec{r}|$, with respect to which that observable $\vec{s}_{\hat{n}} = \hat{n} \cdot \vec{s}$ is measured. Choosing the reference system to be described by a free particle, the measurement of $s_{\hat{n}}$ can be described by the von-Neumann-like interaction, and the total Hamiltonian is:

$$H = \frac{\vec{P}}{2M} + H_s + g(t)q\hat{n} \cdot \vec{s}. \tag{1}$$

Here H_s stands for the Hamiltonian of the spin system. The measurement is described by the last term: q is the conjugate to the "output register", P_q, and $g(t)$ is a time dependent coupling constant, which we shall take to satisfy $\int g(t)dt = g_0$. In the impulsive limit, $g(t) \to g_0\delta(t-t_0)$, the above interaction corresponds to the ordinary von-Neumann measurement.

Notice that in the limit of a continuous measurement, for which $g(t) = constant$ in a finite time interval, the von-Neumann interaction term in Eq. (1), has the same form as the well known monopole-like example of a Berry phase [3]. Thus a Berry phase is expected upon rotation of the reference system. But while in Berry's case the interaction is put in "by hand" just to study its consequences, in our case the interaction naturally arises whenever the spin is measured.

As we shall see, the appearance of the Berry phase and the associated vector potential, can be easily obtained by transforming to a quantum reference-frame. There the spin observable becomes directly measurable and the back-reaction felt by the reference particle is precisely given by the requisite Berry vector potential.

Let us consider first the 2-dimensional case. The reference axis is given in terms of the unit vector $\hat{n} = \hat{x}\cos\phi + \hat{y}\sin\phi$, and $\hat{n}\cdot\vec{s} = s_x\cos\phi + s_y\sin\phi$. The last term in the Hamiltonian (1) above, can be simplified by transforming to a new set of canonical coordinates. The the unitary transformation:

$$U_{(2d)} = e^{-i\phi(s_z-\frac{1}{2})}. \tag{2}$$

yields the relations:

$$p'_\phi = U^\dagger p_\phi U = p_\phi - (s_z - 1/2) \;\; ; \;\; p'_r = p_r \;\; ; \;\; \vec{r}\,' = \vec{r}, \tag{3}$$

$$s'_x = s_x\cos\phi + s_y\sin\phi \;\; ; \;\; s'_y = s_y\cos\phi - s_x\sin\phi \;\; ; \;\; s'_z = s_z. \tag{4}$$

The extra 1/2 factor in (2) is required in order to preserve the single valueness of the wave function, of the combined spin and reference particle system, with respect to the angular coordinate ϕ. (For an integer spin we drop the 1/2).

Expressing the Hamiltonian in terms of the new coordinates we obtain:

$$H = \frac{1}{2M}\left(\vec{p}\,' + \frac{s'_z - 1/2}{r}\hat{\phi}\right)^2 + H'_s + g(t)qs'_x. \tag{5}$$

In the rotated coordinate system, the measuring device interacts directly with s'_x. Conservation of the total angular momentum, $p'_\phi - 1/2$, is guaranteed since $[s'_x, p'_\phi] = 0$. We notice however that in the new coordinate system the reference particle sees the effective vector potential

$$\vec{A}_{(2d)} = \frac{s'_z - 1/2}{r}\hat{\phi}. \tag{6}$$

The latter describes the back-reaction on the reference frame, which here takes the form of a fictitious magnetic fluxon at the origin $\vec{r} = 0$, with a magnetic flux $\Phi = s'_z - 1/2$ in the $\hat{z}$-direction. In the absence of the measurement, $(g(t) = 0)$, the s'_z component of the spin is a constant of motion. Thus, the $2s + 1$ components of the wave function in the s'_z representation evolve independently. The vector potential corresponding to the $s'_z = m_s$ component is $\vec{A}_{(2d)} = (m_s - 1/2)\hat{\phi}/r$, i.e. it corresponds to an integer number of quantized fluxons. Since for all the components the vector potential is equivalent to a pure gauge transformation, and causes no observable effect on the reference particle. On the other hand, during the measurement, the interaction with the measuring device causes a rotation of s_z. In particular, consider the limit of large g_0 and M. In this case we can regard the rapid motion of the spin as following adiabatically the slow motion of the reference particle. The effective vector potential seen by the reference system can therefore be obtained by taking the expectation value of $\vec{A}_{(2d)}$ with respect to the spin wave function:

$$\langle \vec{A}_{(2d)} \rangle = \langle \frac{s_z - 1/2}{r}\hat{\phi} \rangle \approx \frac{1/2}{r}\hat{\phi}. \tag{7}$$

This corresponds to a semi-quantized fluxon at the origin $r = 0$, pointing to the $\hat{z}$ direction. The total phase accumulated in a cyclic motion of the reference particle around the semi-fluxon yields the topological (path independent) phase:

$$\gamma_n = \oint A_{(2d)} dl = n\pi, \tag{8}$$

where n is the winding number.

In the more general case, of a free reference particle in 3-dimensions, the appropriate transformation which maps: $s'_x = U^{\dagger}_{(3d)} s_x U_{(3d)} = \hat{n} \cdot \vec{s}$ is

$$U_{(3d)} = e^{-i(\theta - \pi/2)s_y} e^{-i\phi(s_z - 1/2)}, \tag{9}$$

where θ and ϕ are spherical angles. [4]

The corresponding 3-dimensional vector-potential is in this case

$$A_{(3d)x} = -s_y \frac{\cos\theta\cos\phi}{r} + \left(s_z \sin\theta + s_x \cos\theta - 1/2\right) \frac{\sin\phi}{r\sin\theta}, \tag{10}$$

$$A_{(3d)y} = -s_y \frac{\cos\theta\sin\phi}{r} - \left(s_z \sin\theta + s_x \cos\theta - 1/2\right) \frac{\cos\phi}{r\sin\theta}, \tag{11}$$

$$A_{(3d)z} = s_y \frac{\sin\theta}{r}. \tag{12}$$

For the case of an integer spin or angular momentum the 1/2 above is omitted. It can be verified that $\vec{A}_{(3d)}$ corresponds to a pure gauge non-Abelian vector potential. The field strength vanishes locally, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] = 0$. Thus the force on the reference particle vanishes. Furthermore since the loop integral, $\oint \vec{A}_{(3d)} \cdot d\vec{r}$, gives rise to a trivial flux $2n\pi$, the manifold is simply connected. (This can be seen by noticing that the magnetic field, $\nabla \times \vec{A}_{(3d)}$, due to the terms proportional to s_y vanishes. The other terms correspond to a fluxon pointing in the $\hat{z}$-direction with total flux $\Phi = s_z \sin\theta + s_x \cos\theta - 1/2$ which is quantized for spin components along the direction $\pi/2 - \theta$.) Thus, as in the 2-d case, in absence of coupling with the measuring device, $\vec{A}_{(3d)}$ is a pure gauge vector potential.

In the adiabatic limit discussed above, during the measurement we have $\langle s_z \rangle \approx \langle s_y \rangle \approx 0$. The effective vector potential seen by the reference particle

$$\langle \vec{A}_{(3d)} \rangle = \left(s_x \cos\theta - 1/2\right) \frac{\hat{\phi}}{r\sin\theta}, \tag{13}$$

is identical to the (asymptotical, $r \to \infty$) non-Abelian 't Hooft - Polyakov monopole [5] in the unitary gauge. The effective magnetic field, $\nabla \times \langle \vec{A}_{(3d)} \rangle$:

$$\langle \vec{B} \rangle = s_x \frac{\vec{r}}{r^3} \tag{14}$$

corresponds to that of a magnetic monopole at the origin, $r = 0$, with a magnetic charge $m = s_x$.

The topological vector potential obtained above, clearly have an observable manifestation. Upon rotation of the reference particle around the $\hat{z}$ axis, the particle accumulates an Aharonov-Bohm phase:

$$\gamma_n = \oint \vec{A}_{(3d)} \cdot d\vec{r} = -n\pi(1 - \cos\theta), \tag{15}$$

which equals half of the solid angle subtended by the path. The latter can be observable by means of a standard interference experiment. We thus conclude that during a continuous measurement the back-reaction on the reference particle takes the form of a topological vector potential, of a semi-fluxon in 2-dimensions and that of a monopole in 3-dimensions.

Our discussion above can also be restated in terms of Berry phases. Viewing the reference particle as a slowly changing environment, and the spin system as a fast system which is driven by a time dependent 'environment', we can use the Born-Oppenheimer procedure to solve for the spin's eigenstates. Let us assume for simplicity that g_0 is sufficiently large so H_s can be neglected, and that $g(t)$ is roughly constant. Considering for simplicity the 2-dimensional case, the appropriate eigenstate equation therefore reads

$$gq\hat{n}(\phi) \cdot \vec{s}|\psi(\phi)\rangle = E|\psi(\phi)\rangle, \tag{16}$$

where ϕ is here viewed as the external parameter. For simplicity let us consider the case of $s = 1/2$. We obtain:

$$|\psi_{\pm}(\phi)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\phi/2}|\uparrow_z\rangle \pm e^{+i\phi/2}|\downarrow_z\rangle\right) \otimes |q\rangle. \tag{17}$$

The eigenfunctions $|\psi_\pm\rangle$ are double-valued in the angle ϕ. Thus a cyclic motion in space, which changes ϕ by 2π, induces a sign change. The latter is due to the 'spinorial nature' of fermionic particles, which as is well known, flips sign under a 2π rotation [6]. To obtain the appropriate Berry phase we need to construct single valued solutions of Eq. (17):

$$|\Psi(\phi)\rangle = e^{-i\phi(s_z+1/2)}|\uparrow_x\rangle = \left(e^{-i\phi}|\uparrow_z\rangle \pm |\downarrow_z\rangle\right) \otimes |q\rangle. \tag{18}$$

It then follows that the Berry phase [3]:

$$\gamma_{Berry} = i\oint \langle\Psi(\phi)|\frac{\partial_\phi}{r}|\Psi(\phi)\rangle d\phi = \gamma_n, \tag{19}$$

is identical to the phase, (8), which is induced by the effective semi-fluxon. Similarly, the Berry phase in the 3-dimensional case corresponds to half of the solid angle subtended by the path of the reference particle.

In conclusion, we have shown that the quantum mechanical back-reaction during a measurement induces in certain cases a topological vector potential. The Berry phase can be viewed in this framework, as a necessary consequence of the "strong" nature of the quantum back-reaction.

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[4] In the operator $e^{-i\phi(s_z-1/2)}$ describing the rotation by the spherical angle ϕ, a factor of $-1/2$ is added to the spin operator s_z in order to insure the single-valuedness of the transformed wave-function. On the other hand no such factor needs to be added to the spin operator s_y in the corresponding operator $e^{-i(\theta-\pi/2)s_y}$ which describes the rotation by the $\theta - \pi/2$ spherical angle. The reason is that while the angle ϕ takes values in between $[0, 2\pi]$, the angle θ takes values only in between $[0, \pi]$.

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